Universal C^*-algebraic quantum groups arising from algebraic quantum groups.

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Abstract

In this paper, we construct a universal C*-algebraic quantum group out of an algebraic one. We show that this universal C*-algebraic quantum has the same rich structure as its reduced companion (see [9]). This universal C*-algebraic quantum group also satisfies an upcoming definition of Masuda, Nakagami & Woronowicz except for the possible non-faithfulness of the left Haar weight.

Introduction.

In 1979, Woronowicz proposed the use of the C*-algebra language in the field of quantum groups [32]. Since then, a lot of work has been done in this area but there is still no satisfactory definition of a quantum group in the C*-algebra framework.
However, the following subjects are better understood:

- compact & discrete quantum groups and their duality theory
- some examples: quantum $SU(2)$, quantum Heisenberg, quantum $E(2)$, quantum Lorentz, duals of locally compact groups,...

A good definition of a quantum group should satisfy the following requirements:

- It should incorporate all the known examples and well understood parts of the theory.
- There will have to be a good balance between the theory which can be extracted from the definition and the non complexity of the definition.
- The definition has to allow a consistent duality theory.

The most difficult part of finding a satisfactory axiom scheme for quantum groups seems to be the ability to prove the existence and uniqueness of a left Haar weight from the proposed definition.
At the moment, Masuda, Nakagami and Woronowicz are working on a quasi-final definition of a quantum group in the C*-algebra framework (see also [15]). It is quasi-final in the sense that there is some hope to find simpler axioms which imply their current definition. Later, we will give a preview of this definition.

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In [27], A. Van Daele introduced the notion of Multiplier Hopf-algebras. They are natural generalizations of Hopf algebras to the case of non-unital algebras. Recently, A. Van Daele has looked into the case where such a Multiplier Hopf algebra possesses a non-zero left invariant functional (Haar functional) and found some very interesting properties [23]. It should be noted that everything in this theory is of an algebraic nature.
This category of Multiplier Hopf algebras with a Haar functional behaves very well in different ways:

- This category includes the compact & discrete quantum groups.
- It is possible to construct the dual within this category.
- The category is closed under the double construction of Drinfeld (see [7]).

The last and the first property imply that this category contains the Lorentz quantum group [17]. However, this new category will not exhaust all quantum groups. It is not so difficult to find many classical groups which are not Multiplier Hopf algebras. Also, quantum $E(2)$ will not fit in this scheme. Nevertheless, we still have a nice class of algebraic quantum groups.

In [9], we constructed a reduced C*-algebraic quantum group (in the sense of Masuda, Nakagmi & Woronowicz) out of a Multiplier Hopf$\,^*$-algebra which possesses a positive left invariant functional.
The main purpose of this paper is the construction of a universal C*-algebraic quantum group (in the sense of Masuda, Nakagmi & Woronowicz) out of such a Multiplier Hopf$\,^*$-algebra. The resulting universal C*-algebraic quantum group fits into the definition of Masuda, Nakagali & Woronowicz except for the possible non-faithfulness of the left Haar weight. We will see that this universal C*-algebraic quantum group has the same rich structure as its reduced companion. In order to prove this, we use the results about the algebraic quantum group as well as the results about the reduced C*-algebraic quantum group.

It will probably also be possible to construct some sort of universal quantum group out of an C*-algebraic quantum group according to Masuda, Nakagami & Woronowicz. We believe that this paper will give some ideas how to go around this in the general case. However, some algebraic arguments will have to be modified to analytical ones.

In a first section, we wil give an overview of the results of A. Van Daele about such a Multiplier Hopf$\,^*$-algebra (A, Δ). In a second section, we recapitulate some results concerning the reduced C*-algebraic quantum group (A_r, Δ_r) arising from it. However, most of the results concerning this reduced C*-algebraic quantum group will be introduced when we need them.

In the third section, we introduce the universal C*-algebra A_u together with the universal comultiplication Δ_u. From there on, we gradually prove that this pair (A_u, Δ_u) almost in the scheme of Masuda, Nakagami & Woronowicz (except for the faithfulness of the left Haar weight and some minor detail).

In section 6, we prove that every unitary corepresentation of the reduced quantum group (A_r, Δ_r) is of an algebraic nature. The same is also true for the unitary corepresentations of the universal quantum group (A_u, Δ_u).
The results of section 7 imply that there is a natural bijection between the unitary corepresentations of (A_r, Δ_r) and of (A_u, Δ_u). The same is true for bi-C*-isomorphisms on the reduced and universal quantum group (section 6 and 11).

In [23],A. Van Daele constructs the dual algebraic quantum group $(\hat{A}, \hat{\Delta})$ of (A, Δ). We can in the same way get a universal C*-algebraic quantum group $(\hat{A}_u, \hat{\Delta}_u)$ out of $(\hat{A}, \hat{\Delta})$.
In section 13, we introduce the universal corepresentation of the universal quantum group (A_u, Δ_u) and use it to get a bijective correspondence between unitary corepresentations of (A_u, Δ_u) and non-degenerate *-homomorphisms on the universal C*-algebra $\hat{A}_u$.

Section 14 fixes some notations about weights and gives a thorough overview of slice weights and is necessary to understand some of the results of this paper.

The notations of this paper will be a little bit different from the ones in [9]. Mainly because we have two C*-algebraic quantum groups. As a rule, objects associated to the reduced C*-algebraic quantum group will get the subscript 'r', whereas objects associated to the universal C*-algebraic quantum group will get the subscript 'u'. Objects without a subscript will be connected to the algebraic quantum group.

First, we introduce some notations and conventions. We will always use the minimal tensor product between C*-algebras and use the symbol $\otimes$ for this complete tensor product. For any C*-algebra A, we denote the Multiplier C*-algebra by $M(A)$. The flip map between two C*-algebras will be denoted by χ. For the algebraic tensor products of vector spaces and linear mappings, we use the symbol $\odot$. The algebraic dual of a vector space V will be denoted by V'.
Let H be a Hilbert space. Then $B(H)$ will denote the C*-algebra of bounded operators on H, whereas $B_0(H)$ will denote the C*-algebra of compact operators on H. Consider vectors $v, w \in H$, then $\omega_{v,w}$ is the element in $B_0(H)^*$ such that $\omega_{v,w}(x) = \langle xv, w \rangle$ for all $x \in B_0(H)$.
The domain of an unbounded operator T on H is denoted by $D(T)$.
The domain of an element α which is affiliated with some C* , will be denoted by $\mathcal{D}(\alpha)$ (we will use the same notation for closed mappings in a C* which arise from one-parameter groups).
Whenever we say that an unbounded operator is positive, it is included that this operator is also selfadjoint. The same rules apply to elements affiliated with a C*-algebra. Let α be an element affiliated with a C* A, then α is called strictly positive if α is positive and has dense range (it is then automatically injective).
A one-parameter group σ on a C* A is called norm-continuous if and only if for every $a \in A$, the mapping $\mathbb{R} \to A : t \mapsto \sigma_t(a)$ is norm-continuous.

For a very readable introduction to Hilbert-C*-modules, we refer to [14]. Let E and F be two Hilbert-C*-modules over a C*-algebra A. Then the set of adjointable mappings from E into F will be denoted by $\mathcal{L}(E, F)$ whereas the set of compact operators from E into F will be denoted by $\mathcal{K}(E, F)$.
For the notion of regular operators between to Hilbert-C*-modules, we refer to [14].

As promised, we give now a preview of the definition of a C*-algebraic quantum group according to Masuda, Nakagami & Woronowicz:

Let B be a C*-algebra and Δ a non-degenerate *-homomorphism from B into $M(B \otimes B)$ such that

1. Δ is coassociative, i. e. $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$.
2. Δ satisfies the following density conditions: $\Delta(B)(B \otimes 1)$ and $\Delta(B)(1 \otimes B)$ are dense subsets of $B \otimes B$.

Furthermore, we assume the existence of the following objects:

1. a KMS-weight φ on B with modular group σ,
2. a norm continuous one parameter group τ on B,
3. an involutive *-anti-automorphism R on B,

which satisfy the following properties:

1. For every $a \in \mathcal{M}_\varphi$, we have that $\Delta(a)$ belongs to $\overline{\mathcal{M}}_{\iota\otimes\varphi}$ and $(\iota \otimes \varphi)\Delta(a) = \varphi(a)1$.

2. Consider $a, b \in \mathcal{N}_\varphi$. Let $\omega \in B^*$ such that $\omega R \tau_{-\frac{i}{2}}$ is bounded and call θ the unique element in B^* which extends $\omega R \tau_{-\frac{i}{2}}$. Then

$$\varphi(\, b^* \, (\omega \otimes \iota)\Delta(a)\,) = \varphi(\, (\theta \otimes \iota)(\Delta(b^*))\, a\,) \; .$$

3.
   - φ is invariant under τ.
   - φ commutes with φR.

4.
   - For every $t \in \mathbb{R}$, we have that $R\tau_t = \tau_t R$.
   - We have that $\Delta \tau_t = (\tau_t \otimes \tau_t)\Delta$ for all $t \in \mathbb{R}$.
   - $\Delta R = \chi(R \otimes R)\Delta$

Then, we call $(B, \Delta, \varphi, \tau, R)$ a C*-algebraic quantum group.

We call φ the left Haar weight, R the anti-unitary antipode and τ the scaling group of our quantum group. We put $\kappa = R\tau_{-\frac{i}{2}}$, then κ plays the role of the antipode of our quantum group.
This definition seems to be a C*-version of a definition of a quantum group in the von Neumann algebra setting (see [15]), which in turn was a generalization of the (too restrictive) definition of a Kac-algebra (see [6]). We are not sure that this will be the ultimate definition of a C*-algebraic quantum group proposed by Masuda, Nakagami & Woronowicz, but we expect that this one gives a fairly good idea of it.
A possible drawback of this definition is the complexity of the axioms. However, we will show that the C*-algebraic versions of Van Daele's algebraic objects fit almost in this scheme. The only difference lies in the fact that we only can prove that φ is relatively invariant with respect to τ in stead of invariant. It is not clear at the moment whether this definition of Masuda, Nakagami & Woronowicz should be modified in this respect.

1 Algebraic quantum groups

In this first section, we will introduce the notion of an algebraic quantum group as can be found in [23]. Moreover, we will give an overview of the properties of this algebraic quantum group. The proofs of these results can be found in the same paper [23]. After this section, we will construct a universal C*-algebraic quantum group out of this algebraic one, thereby heavily depending on the material gathered in this section and [9]. We will first introduce some terminology.

We call a *-algebra A non-degenerate if and only if we have for every $a \in A$ that :

$$(\forall b \in A : ab = 0) \Rightarrow a = 0 \qquad\qquad \text{and} \qquad\qquad (\forall b \in A : ba = 0) \Rightarrow a = 0.$$

For a non-degenerate *-algebra A, you can define the multiplier algebra $M(A)$. This is a unital *-algebra in which A sits as a selfadjoint ideal (the definition of this multiplier algebra is the same as in the case of C*-algebras).

If you have two non-degenerate *-algebras A, B and a multiplicative linear mapping π from A to $M(B)$, we call π non-degenerate if and only if the vector spaces $\pi(A)B$ and $B\pi(A)$ are equal to B. Such a non-degenerate multiplicative linear map has a unique multiplicative linear extension to $M(A)$. This extension will be denoted by the same symbol as the original mapping. Of course, we have similar definitions and results for antimultiplicative mappings. If we work in an algebraic setting, we will always use this form of non-degeneracy as opposed to the non degeneracy of *-homomorphisms between C*-algebras!

For a linear functional ω on a non-degenerate *-algebra A and any $a \in M(A)$ we define the linear functionals ωa and $a\omega$ on A such that $(a\omega)(x) = \omega(xa)$ and $(\omega a)(x) = \omega(ax)$ for every $x \in A$.

You can find some more information about non-degenerate algebras in the appendix of [27].

Let ω be a linear functional on a *-algebra A, then :

1. ω is called positive if and only if $\omega(a^*a)$ is positive for every $a \in A$.
2. If ω is positive, then ω is called faithful if and only if for every $a \in A$, we have that

$$\omega(a^*a) = 0 \Rightarrow a = 0.$$

We have now gathered the necessary information to understand the following definition

Definition 1.1 *Consider a non-degenerate *-algebra A and a non-degenerate *-homomorphism Δ from A into $M(A \odot A)$ such that*

1. $(\Delta \odot \iota)\Delta = (\iota \odot \Delta)\Delta$.
2. *The linear mappings T_1, T_2 from $A \odot A$ into $M(A \odot A)$ such that*

$$T_1(a \otimes b) = \Delta(a)(b \otimes 1) \qquad \textit{and} \qquad T_2(a \otimes b) = \Delta(a)(1 \otimes b)$$

for all $a, b \in A$, are bijections from $A \odot A$ to $A \odot A$.

Then we call (A, Δ) a Multiplier Hopf*-algebra.

In [27], A. Van Daele proves the existence of a unique non-zero *-homomorphism ε from A to $\mathbb{C}$ such that

$$(\varepsilon \odot \iota)\Delta = (\iota \odot \varepsilon)\Delta = \iota \ .$$

Furthermore, he proves the existence of a unique anti-automorphism S on A such that

$$m(S \odot \iota)(\Delta(a)(1 \otimes b)) = \varepsilon(a)b \qquad \text{and} \qquad m(\iota \odot S)((b \otimes 1)\Delta(a)) = \varepsilon(a)b$$

for every $a, b \in A$ (here, m denotes the multiplication map from $A \odot A$ to A). As usual, ε is called the counit and S the antipode of (A, Δ). Moreover, $S(S(a^*)^*) = a$ for all $a \in A$. Also, $\chi(S \odot S)\Delta = \Delta S$.

Let ω be a linear functional on A an a an element in A. We define the element $(\omega \odot \iota)\Delta(a)$ in $M(A)$ such that

- $(\omega \odot \iota)(\Delta(a))\ b = (\omega \odot \iota)(\Delta(a)(1 \otimes b))$
- $b\ (\omega \odot \iota)(\Delta(a)) = (\omega \odot \iota)((1 \otimes b)\Delta(a))$

for every $b \in A$.
In a similar way, the multiplier $(\iota \odot \omega)\Delta(a)$ is defined.

Let ω be a linear functional on A. We call ω left invariant (with respect to (A, Δ)), if and only if $(\iota \odot \omega)\Delta(a) = \omega(a)\,1$ for every $a \in A$. Right invariance is defined in a similar way.

Definition 1.2 *Consider a Multiplier Hopf*-algebra (A, Δ) such that there exists a non-zero positive linear functional φ on A which is left invariant. Then we call (A, Δ) an algebraic quantum group.*

For the rest of this paper, we will fix an algebraic quantum group (A, Δ) together with a non-zero left invariant positive linear functional φ on it.
An important feature of such an algebraic quantum group is the faithfulness and uniqueness of left invariant functionals :

1. Consider a left invariant linear functional ω on A, then there exists a unique element $c \in \mathbb{C}$ such that $\omega = c\,\varphi$.

2. Consider a non-zero left invariant linear functional ω on A, then ω is faithful.

In particular, φ is faithful.

A first application of this uniqueness result concerns the antipode : Because φS^2 is left invariant, there exists a unique complex number μ such that $\varphi S^2 = \mu \varphi$ (in [23], our μ is denoted by τ!). It is not so difficult to prove in an algebraic way that $|\mu| = 1$. The question remains open if there exists an example of an algebraic quantum group (in our sense) with $\mu \neq 1$.

It is clear that φS is a non-zero right invariant linear functional on A. However, in general, φS will not be positive. In [9], we use the C*-algebra approach to prove the existence of a non-zero positive right invariant linear functional on A.
Of course, we have similar faithfulness and uniqueness results about right invariant linear functionals.

In this paper, we will need frequently the following formula :

$$(\iota \odot \varphi)(\,(1 \otimes a)\Delta(b)\,) = S(\,(\iota \odot \varphi)(\Delta(a)(1 \otimes b))\,) \tag{1}$$

for all $a, b \in A$. A proof of this result can be found in proposition 3.11 of [23]. It is in fact nothing else but an algebraic form of the strong left invariance in the definition of Masuda, Nakagami & Woronowicz.

Another non-trivial property about φ is the existence of a unique automorphism ρ on A such that $\varphi(ab) = \varphi(b\rho(a))$ for every $a, b \in A$. We call this the weak KMS-property of φ (In [23], our mapping ρ is denoted by σ!).
This weak KMS-property is crucial to extend φ to a weight on the C*-algebra level. We have moreover that $\rho(\rho(a^*)^*) = a$ for every $a \in A$.
As usual, there exists a similar object ρ' for the right invariant functional φS, i.e. ρ' is an automorphism on A such that $(\varphi S)(ab) = (\varphi S)(b\rho'(a))$ for every $a, b \in A$.
Using the antipode, we can connect ρ and ρ' via the formula $S\rho' = \rho S$. Furthermore, we have that S^2 commutes with ρ and ρ'. The interplay between ρ,ρ' and Δ is given by the following formulas :

$$\Delta\rho = (S^2 \odot \rho)\Delta \qquad \text{and} \qquad \Delta\rho' = (\rho' \odot S^{-2})\Delta.$$

It is also possible to introduce the modular function of our algebraic quantum group. This is an invertible element δ in $M(A)$ such that

$$(\varphi \odot \iota)(\Delta(a)(1 \otimes b)) = \varphi(a)\,\delta\,b$$

for every $a, b \in A$.
Concerning the right invariant functional, we have that

$$(\iota \odot \varphi S)(\Delta(a)(b \otimes 1)) = (\varphi S)(a)\,\delta^{-1}\,b$$

for every $a, b \in A$.
This modular function is, like in the classical group case, a one dimensional (generally unbounded) corepresentation of our algebraic quantum group :

$$\Delta(\delta) = \delta \odot \delta \qquad \varepsilon(\delta) = 1 \qquad S(\delta) = \delta^{-1}.$$

As in the classical case, we can relate the left invariant functional to our right invariant functional via the modular function : we have for every $a \in A$ that

$$\varphi(S(a)) = \varphi(a\delta) = \mu\,\varphi(\delta a).$$

If we apply this equality two times and use the fact that $S(\delta) = \delta^{-1}$, we get that $\varphi(S^2(a)) = \varphi(\delta^{-1} a \delta)$ for every $a \in A$.

Not surprisingly, we have also that $\rho(\delta) = \rho'(\delta) = \mu^{-1}\delta$.

Another connection between ρ and ρ' is given by the equality $\rho'(a) = \delta\rho(a)\delta^{-1}$ for all $a \in A$.

We have also a property which says, loosely speaking, that every element of A has compact support : Consider $a_1, \ldots, a_n \in A$. Then there exists an element c in A such that $c\, a_i = a_i\, c = a_i$ for every $i \in \{1, \ldots, n\}$.

In a last part, we are going to say something about duality.
We define the subspace $\hat{A}$ of A' as follows :

$$\hat{A} = \{\, \varphi a \mid a \in A \,\} = \{\, a\varphi \mid a \in A \,\}.$$

Like in the theory of Hopf*-algebras, we turn $\hat{A}$ into a non-degenerate *-algebra :

1. For every $\omega_1, \omega_2 \in \hat{A}$ and $a \in A$, we have that $(\omega_1\omega_2)(a) = (\omega_1 \odot \omega_2)(\Delta(a))$.
2. For every $\omega \in \hat{A}$ and $a \in A$, we have that $\omega^*(a) = \overline{\omega(S(a)^*)}$.

We should remark that a little bit of care has to be taken by defining the product and the *-operation in this way.
Also, a comultiplication $\hat{\Delta}$ can be defined on $\hat{A}$ such that $\hat{\Delta}(\omega)(x \otimes y) = \omega(xy)$ for every $\omega \in \hat{A}$ and $x, y \in A$.
Again, this has to be made more precise. This can be done by embedding $M(A \odot A)$ into $(A \odot A)'$ in the right way but we will not go into this subject (see [11] for more information about this). A definition of the comultiplication $\hat{\Delta}$ without the use of such an embedding can be found in definition 4.4 of [23].
In this way, $(\hat{A}, \hat{\Delta})$ becomes a Multiplier Hopf*-algebra. The counit $\hat{\varepsilon}$ and the antipode $\hat{S}$ are such that

1. For every $\omega \in \hat{A}$, we have that $\hat{\varepsilon}(\omega) = \omega(1)$.
2. For every $\omega \in \hat{A}$ and every $a \in A$, we have that $\hat{S}(\omega)(a) = \omega(S(a))$.

For any $a \in A$, we define $\hat{a} = a\varphi \in \hat{A}$. The mapping $A \to \hat{A} : a \mapsto \hat{a}$ is a bijection, which is in fact nothing else but the Fourier transform.

Next, we define the linear functional $\hat{\psi}$ on $\hat{A}$ such that $\hat{\psi}(\hat{a}) = \varepsilon(a)$ for every $a \in A$. It is possible to prove that $\hat{\psi}$ is right invariant.
Also, we have that $\hat{\psi}(\hat{a}^*\hat{a}) = \varphi(a^*a)$ for every $a \in A$. This implies that $\hat{\psi}$ is a non-zero positive right invariant linear functional on $\hat{A}$.

From theorem 9.9 of [9], we know that (A, Δ) possesses a non-zero positive right invariant linear functional. In a similar way, this functional will give rise to a non-zero positive left invariant linear functional on $\hat{A}$. This will imply that $(\hat{A}, \hat{\Delta})$ is again an algebraic quantum group.

In definition 6.5 of [11], we introduced the universal corepresentation X of A (it was denoted by U in [11]). This is an element of $M(A \odot \hat{A})$ such that $(\Delta \odot \iota)(X) = X_{13}\, X_{23}$ and $(\iota \odot \hat{\Delta})(X) = X_{12}\, X_{13}$.
This element X serves as a bridge between unitary corepresentations of (A, Δ) and non-degenerate *-homomorphisms on $\hat{A}$.

Consider a unitary corepresentation $\mathcal{U}$ of (A, Δ) on a *-algebra C. Let $\omega \in A'$ and $a \in A$.
By proposition 4.3 and result 5.9, we have an element $(a\omega \odot \iota)(\mathcal{U})$ in $M(C)$ such that the following holds :

1. We have for every $c \in C$ that $\mathcal{U}(a \otimes c)$ belongs to $A \odot C$ and $(a\omega \odot \iota)(\mathcal{U})\, c = (\omega \odot \iota)(\mathcal{U}(a \otimes c))$.

2. We have for every $c \in C$ that $(1 \otimes c)\mathcal{U}(a \otimes 1)$ belongs to $A \odot C$ and $c\,(a\omega \odot \iota)(\mathcal{U}) = (\omega \odot \iota)((1 \otimes c)\mathcal{U}(a \otimes 1))$.

Notice that in this way, we defined $(\rho \odot \iota)(\mathcal{U})$ for every $\rho \in \hat{A}$.

Proposition 1.3 *Consider a non-degenerate *-algebra C. Let $\mathcal{U}$ be a unitary corepresentation of (A, Δ) on C. Define the mapping θ from $\hat{A}$ into $M(C)$ such that $\theta(\omega) = (\omega \odot \iota)(\mathcal{U})$ for every $\omega \in \hat{A}$. Then θ is a non-degenerate *-homomorphism from $\hat{A}$ into $M(C)$ such that $\theta(a\omega) = (a\omega \odot \iota)(\mathcal{U})$ for every $a \in A$ and $\omega \in A'$.*

Theorem 1.4 *Consider a non-degenerate *-algebra C. Let $\mathcal{U}$ be a unitary corepresentation of (A, Δ) on C. Define the mapping θ from $\hat{A}$ into $M(C)$ such that $\theta(\omega) = (\omega \odot \iota)(\mathcal{U})$ for every $\omega \in \hat{A}$. Then θ is the unique non-degenerate *-homomorphism from $\hat{A}$ into $M(C)$ such that $(\theta \odot \iota)(X) = \mathcal{U}$.*

We have also a converse of this :

Theorem 1.5 *Consider a non-degenerate *-algebra C. Let θ be a non-degenerate *-homomorphism of $\hat{A}$ into $M(C)$. Define $\mathcal{U} = (\iota \odot \theta)(X)$. Then $\mathcal{U}$ is the unique unitary corepresentation of (A, Δ) on C satisfying $\theta(\rho) = (\rho \odot \iota)(\mathcal{U})$ for every $\rho \in \hat{A}$.*

2 The reduced C*-algebraic quantum group arising from (A, Δ)

In [9], we constructed a reduced C*-algebraic quantum group (A_r, Δ_r) in the sense of Masuda, Nakagami & Woronowicz out of the algebraic quantum group (A, Δ). The aim of this paper to construct the universal C*-algebraic quantum group out of (A, Δ). We will show that this resulting universal C*-algebraic quantum group has the same rich structure as its reduced companion.
The proofs of the results about the universal case depend heavily on the results concerning the reduced case. Therefore we first recapitulate some results about the reduced C*-algebraic quantum group. This section gives an overview of the results of section 2 of [9] .

For the rest of this paper, we fix a GNS-pair (H, Λ) of the left Haar functional φ on A. This means that H is a Hilbert space and Λ is a linear mapping from A into H such that

1. The set $\Lambda(A)$ is dense in H.

2. We have for every $a, b \in A$ that $\langle \Lambda(a), \Lambda(b) \rangle = \varphi(b^* a)$.

As usual, we can associate a multiplicative unitary to (A, Δ) :

Definition 2.1 *We define W as the unique unitary element in $B(H \otimes H)$ such that $W\,(\Lambda \odot \Lambda)(\Delta(b)(a \otimes 1)) = \Lambda(a) \otimes \Lambda(b)$ for every $a, b \in A$. The element W is called the fundamental unitary associated to (A, Δ).*

The coassociativity of Δ on the algebra level implies that W is multiplicative : $W_{12}W_{13}W_{23} = W_{23}W_{12}$.

The GNS-pair (H, Λ) allows us to represent A by bounded operators on H :

Definition 2.2 *We define π_r as the unique *-homomorphism from A into $B(H)$ such that $\pi_r(a)\Lambda(b) = \Lambda(ab)$ for every $a, b \in A$. We have also that π_r is injective.*

We would like to mention that π_r is denoted by π in [9], but we will reserve the symbol π for another object in this paper!

Notice also that it is not immediate that $\pi_r(x)$ is a bounded operator on H (because φ is merely a functional, not a weight), but the boundedness of $\pi_r(x)$ is connected with the following equality :

We have for every $a, b \in A$ that

$$\pi_r\big((\iota \otimes \varphi)(\Delta(b^*)(1 \otimes a))\big) = (\iota \otimes \omega_{\Lambda(a),\Lambda(b)})(W) \tag{2}$$

The mapping π_r makes it possible to define our reduced C*-algebra :

Definition 2.3 *We define A_r as the closure of $\pi_r(A)$ in $B(H)$. So A_r is a non-degenerate sub-C*-algebra of $B(H)$.*

Equation 2 implies that

$$A_r = \text{ closure of } \{ (\iota \otimes \omega)(W) \mid \omega \in B_0(H)^* \} \text{ in } B(H) . \tag{3}$$

As usual, we use the fundamental unitary to define a comultiplication on A_r. We will denote it by Δ_r (in [9], it is denoted by Δ).

Definition 2.4 *We define the mapping Δ_r from A_r into $B(H \otimes H)$ such that $\Delta_r(x) = W^*(1 \otimes x)W$ for all $x \in A_r$. Then Δ_r is an injective *-homomorphism.*

It is not so difficult to show that Δ_r on the C*-algebra level is an extension of Δ on the *-algebra level :

Result 2.5 *We have for all $a \in A$ and $x \in A \odot A$ that $(\pi_r \odot \pi_r)(x)\, \Delta_r(\pi_r(a)) = (\pi_r \odot \pi_r)(x\, \Delta(a))$ and $\Delta_r(\pi_r(a))\, (\pi_r \odot \pi_r)(x) = (\pi_r \odot \pi_r)(\Delta(a)\, x)$.*

Using this result, it is easy to prove formulas like :

$$\Delta_r(\pi_r(a))\,(1 \otimes \pi_r(b)) = (\pi_r \odot \pi_r)(\Delta(a)(1 \otimes b))$$

for all $a, b \in A$.

Using the above results, it is not so hard to prove the following theorem :

Theorem 2.6 *We have that A_r is a non-degenerate sub-C*-algebra of $B(H)$ and Δ_r is a non-degenerate injective *-homomorphism from A_r into $M(A_r \otimes A_r)$ such that :*

1. *$(\Delta_r \otimes \iota)\Delta_r = (\iota \otimes \Delta_r)\Delta_r$*
2. *The vector spaces $\Delta_r(A_r)(A_r \otimes 1)$ and $\Delta_r(A_r)(1 \otimes A_r)$ are dense subspaces of $A_r \otimes A_r$.*

It is also possible to represent the dual multiplier Hopf *-algebra $\hat{A}$ on H.
Remember from section 1 that we have a non-zero positive right invariant linear functional $\hat{\psi}$ on $\hat{A}$. We have moreover that $\hat{\psi}(\hat{b}^*\, \hat{a}) = \varphi(b^* a)$ for all $a, b \in A$.

We define the linear map $\hat{\Lambda}$ from $\hat{A}$ into H such that $\hat{\Lambda}(\hat{a}) = \Lambda(a)$ for every $a \in A$. Then

1. $\hat{\Lambda}(\hat{A})$ is dense in H

2. $\langle \hat{\Lambda}(a), \hat{\Lambda}(b) \rangle = \hat{\psi}(b^* a)$ for every $a, b \in \hat{A}$.

The multiplicative unitary W can also be expressed in terms of $\hat{\Lambda}$:

Result 2.7 *We have for every $\omega_1, \omega_2 \in \hat{A}$ that $W(\hat{\Lambda}(\omega_1) \otimes \hat{\Lambda}(\omega_2)) = (\hat{\Lambda} \odot \hat{\Lambda})(\hat{\Delta}(\omega_1)(1 \otimes \omega_2))$*

With this expression in hand, we can do the same things for the dual $\hat{A}$ as we did for A itself.

Definition 2.8 *We define the mapping $\hat{\pi}_r$ from $\hat{A}$ into $B(H)$ such that $\hat{\pi}_r(\omega)\hat{\Lambda}(\theta) = \hat{\Lambda}(\omega\theta)$ for all $\omega, \theta \in \hat{A}$. Then $\hat{\pi}_r$ is an injective *-homomorphism.*

Furthermore, we have for every $\theta, \eta \in \hat{A}$ that

$$(\omega_{\hat{\Lambda}(\theta),\hat{\Lambda}(\eta)} \otimes \iota)(W) = \hat{\pi}_r\big((\hat{\psi} \odot \iota)((\eta^* \otimes 1)\hat{\Delta}(\theta))\big). \tag{4}$$

Definition 2.9 *We define $\hat{A}_r$ as the closure of $\hat{\pi}_r(\hat{A})$ in $B(H)$. So $\hat{A}_r$ is a non-degenerate sub-C^*-algebra of $B(H)$.*

Equation 4 implies that

$$\hat{A}_r = \quad \text{closure of} \quad \{\, (\omega \otimes \iota)(W) \mid \omega \in B_0(H)^* \,\} \;\text{ in } B(H),$$

which is again something familiar.

Definition 2.10 *We define the mapping $\hat{\Delta}_r$ from $\hat{A}_r$ into $B(H \otimes H)$ such that $\hat{\Delta}_r(x) = W(x \otimes 1)W^*$ for all $x \in \hat{A}_r$. Then $\hat{\Delta}_r$ is an injective *-homomorphism.*

We will of course also have the following theorem.

Theorem 2.11 *We have that $\hat{A}_r$ is a non-degenerate sub-C^*-algebra of $B(H)$ and $\hat{\Delta}_r$ is a non-degenerate injective *-homomorphism from $\hat{A}_r$ into $M(\hat{A}_r \otimes \hat{A}_r)$ such that :*

1. $(\hat{\Delta}_r \otimes \iota)\hat{\Delta}_r = (\iota \otimes \hat{\Delta}_r)\hat{\Delta}_r$
2. *The vector spaces $\hat{\Delta}_r(\hat{A}_r)(\hat{A}_r \otimes 1)$ and $\hat{\Delta}_r(\hat{A}_r)(1 \otimes \hat{A}_r)$ are dense subspaces of $\hat{A}_r \otimes \hat{A}_r$.*

Lemma 2.21 of [9] and definition 6.5 of [11] imply the next proposition.

Proposition 2.12 *We have for every $x \in A_r \odot \hat{A}_r$ that $W(\pi_r \odot \hat{\pi}_r)(x) = (\pi_r \odot \hat{\pi}_r)(X\,x)$ and $(\pi_r \odot \hat{\pi}_r)(x)\,W = (\pi_r \odot \hat{\pi}_r)(x\,X)$.*

Corollary 2.13 *The element W belongs to $M(A_r \otimes \hat{A}_r)$.*

An important object associated to (A_r, Δ_r) is the left Haar weight on A_r. It is determined by the following theorem (see [9], theorem 6.12, proposition 6.2 and the remarks after it) :

Theorem 2.14 *There exists a unique closed linear map Λ_r from within A_r into H such that $\pi_r(A)$ is a core for Λ_r and $\Lambda_r(\pi_r(a)) = \Lambda(a)$ for every $a \in A$.*
There exists moreover a unique weight φ_r on A_r such that (H, Λ_r, ι) is a GNS-construction for φ_r.
We have also that $\pi(A) \subseteq \mathcal{M}_{\varphi_r}$ and that $\varphi_r(\pi(a)) = \varphi(a)$ for every $a \in A$.

Proposition 2.15 *The weight φ_r is a faithful KMS-weight. We denote the modular group of φ_r by σ_r.*

We denote the modular operator of φ_r by ∇ and the modular conjugation of φ_r by J (both with respect to the GNS-representation (H, Λ_r, ι)).

Put $T = J\nabla^{\frac{1}{2}}$. By the remarks after proposition 3.2 of [9], we know that $\Lambda(A)$ is a core for T and that $T\Lambda(a) = \Lambda(a^*)$ for every $a \in A$. We have moreover that $\Lambda(A) \subseteq D(T^*)$ and that $T^*\Lambda(a) = \Lambda(\rho(a^*))$ for every $a \in A$.

So $\Lambda(A) \subseteq D(\nabla)$ and $\nabla\Lambda(a) = \Lambda(\rho(a))$ for every $a \in A$.
This implies for every element $a \in A$ that $\Lambda(a)$ is analytic with respect to ∇ and that $\nabla^n\Lambda(a) = \Lambda(\rho^n(a))$ for every $n \in \mathbb{Z}$.

The left invariance on the *-algebra level is then transferred to the left invariance on the C*-algebra level. For used notations, we refer to the appendix.

Theorem 2.16 *Consider $x \in \mathcal{M}_{\varphi_r}$, then $\Delta_r(x)$ belongs to $\overline{\mathcal{M}}_{\iota\otimes\varphi_r}$ and $(\iota \otimes \varphi_r)\Delta_r(x) = \varphi_r(x)\,1$.*

For some more detailed information about the left invariance, we refer to section 6 of [9] and [12].
We will need the following natural formula for W. For notations, we refer to section 14.2

Proposition 2.17 *Consider $a, b \in \mathcal{N}_{\varphi_r}$. Then $\Delta(b)(a \otimes 1)$ belongs to $\mathcal{N}_{\varphi_r\otimes\varphi_r}$ and $W(\Lambda_r \otimes \Lambda_r)(\Delta(b)(a \otimes 1)) = \Lambda_r(a) \otimes \Lambda_r(b)$.*

This proposition follows from the definition of W and the definition of Λ_r. We will give an explicit proof of a similar result in a later section (see proposition 10.20).

3 The universal bi-C*-algebra

In this section, we will introduce the universal enveloping C*-algebra of the *-algebra A and extend the comultiplication to this C*-algebra. In order to do so, we will need the following crucial lemma.

Lemma 3.1 *Consider a C*-algebra C and a sub-*-algebra B of $M(C)$ such that BC is dense in C. Let ϕ be a *-homomorphism from $\hat{A}$ into $M(B)$ such that $\phi(\hat{A})B = B$.*
Take $\omega \in \hat{A}$ and define θ to be the unique element in A_r^* such that $\theta\circ\pi_r = \omega$. Then there exists a unique element T in $M(C)$ such that $\phi(\omega)\,b = T\,b$ and $b\,\phi(\omega) = b\,T$ for every $b \in B$. Furthermore, $\|T\| \leq \|\theta\|$.

Remark 3.2 We should be careful with the terminology above. The *-algebra B has only an algebraic character and the mapping ϕ is an algebraically non-degenerate *-homomorphism from $\hat{A}$ into $M(B)$. This implies that $\phi(\omega)$ is an element of the algebraic multiplier *-algebra $M(B)$. In general, there is no reason for $\phi(\omega)$ to be an element of $M(C)$. Nevertheless, the previous lemma guarantees that this is true in this special case.

Proof of the lemma : For the moment, consider π_r as an algebraically non-degenerate *-homomorphism from A into $\pi_r(A)$ and ϕ as an algebraically non-degenerate *-homomorphism from $\hat{A}$ into $M(B)$ (The non-degeneracy of π_r follows from the fact that $A^2 = A$).
Then the remarks at the end of section 1 imply the existence of a unitary element $\mathcal{V}$ in the algebraic multiplier *-algebra $M(A \odot B)$ such that $\phi(\rho) = (\rho \odot \iota)(\mathcal{V})$ for every $\rho \in \hat{A}$. Put $\mathcal{P} = (\pi_r \odot \iota)(\mathcal{V})$, then $\mathcal{P}$ is a unitary element in the algebraic multiplier *-algebra $M(\pi_r(A) \odot B)$.

It is easy to see that

$$\big([\mathcal{P}(a_2 \otimes b_2)](1 \otimes c_2)\big)^* \big([\mathcal{P}(a_1 \otimes b_1)](1 \otimes c_1)\big) = (a_2 \otimes b_2 c_2)^* \, (a_1 \otimes b_1 c_1)$$

for every $a_1, a_2 \in \pi_r(A)$, $b_1, b_2 \in B$, and $c_1, c_2 \in C$. Using the fact that $\pi_r(A)$ is dense in A_r and the fact that BC is dense in C, this implies the existence of a unique unitary element $\mathcal{Q} \in M(A_r \otimes C)$ such that

$$\mathcal{Q}(a \otimes bc) = [\mathcal{P}(a \otimes b)](1 \otimes c)$$

for every $a \in \pi_r(A)$, $b \in B$ and $c \in C$.
This implies immediately that $\mathcal{Q}(a \otimes b) = \mathcal{P}(a \otimes b)$ for every $a \in \pi_r(A)$ and $b \in B$.
Define $T = (\theta \otimes \iota)(\mathcal{Q})$, then T is an element in $M(C)$ and $\|T\| \leq \|\theta\|$.

Because ω belongs to $\hat{A}$, there exists $a \in A$ such that $\omega = a\omega$. Because $\theta \circ \pi_r = \omega$, this implies that $\theta = \pi_r(a)\theta$. So we get for every $b \in B$ that

$$\begin{aligned} T\,b &= (\theta \otimes \iota)(\mathcal{Q})\,b = (\pi_r(a)\theta \otimes \iota)(\mathcal{Q})\,b = (\theta \otimes \iota)(\mathcal{Q}(\pi_r(a) \otimes b)) \\ &= (\theta \otimes \iota)(\mathcal{P}(\pi_r(a) \otimes b)) = (\theta \odot \iota)(\mathcal{P}(\pi_r(a) \otimes b)) \\ &= (\theta \odot \iota)((\pi_r \odot \iota)(\mathcal{V})\,(\pi_r(a) \otimes b)) = (\theta \odot \iota)\big((\pi_r \odot \iota)(\mathcal{V}(a \otimes b))\big) \\ &= (\omega \odot \iota)(\mathcal{V}(a \otimes b)) = (a\omega \odot \iota)(\mathcal{V})\,b = (\omega \odot \iota)(\mathcal{V})\,b = \phi(\omega)\,b\ . \end{aligned}$$

Using the associativity in $M(B)$ and $M(C)$, we get from this result also that $b\,\phi(\omega) = b\,T$ for every $b \in B$.
■

Result 3.3 *Consider a C^*-algebra C and a *-homomorphism ϕ from $\hat{A}$ into $M(C)$ such that $\phi(\hat{A})C$ is dense in C. Let ω be an element in $\hat{A}$ and define θ to be the unique element in A_r^* such that $\theta \circ \pi_r = \omega$. Then $\|\phi(\omega)\| \leq \|\theta\|$.*

Proof : Put $B = \phi(\hat{A})$. We have by assumption that BC is dense in C. It is also clear that ϕ is a *-homomorphism from $\hat{A}$ into B. Because $\hat{A}^2 = \hat{A}$, we have that $\phi(\hat{A})B = B$.
Therefore, the previous lemma implies the existence of a unique element $T \in M(C)$ such that $\|T\| \leq \|\theta\|$ and $T\,b = \phi(\omega)\,b$ for every $b \in B$. Because BC is dense in C, this implies that $T = \phi(\omega)$. ■

This result implies the following one :

Corollary 3.4 *Consider a C^*-algebra C and a *-homomorphism ϕ from $\hat{A}$ into C. Let ω be an element in $\hat{A}$ and define θ to be the unique element in A_r^* such that $\theta \circ \pi_r = \omega$. Then $\|\phi(\omega)\| \leq \|\theta\|$.*

As a consequence, we get the following boundedness property.

Corollary 3.5 *Consider $\omega \in \hat{A}$. Then there exists a positive number M such that $\|\phi(\omega)\| \leq M$ for every C^*-algebra C and every *-homomorphism ϕ from $\hat{A}$ into C.*

Another application of lemma 3.1 can be found in the following result :

Result 3.6 *Consider two C^*-algebras C_1, C_2 and *-homomorphisms ϕ_1 from $\hat{A}$ into $M(C_1)$ and ϕ_2 from $\hat{A}$ into $M(C_2)$ such that $\phi_1(\hat{A})C_1$ is dense in C_1 and $\phi_2(\hat{A})C_2$ is dense in C_2.
Then there exists a unique *-homomorphism ϕ from $\hat{A}$ into $M(C_1 \otimes C_2)$ such that $\phi(\omega)\,(\phi_1(\omega_1) \otimes \phi_2(\omega_2)) = (\phi_1 \odot \phi_2)(\hat{\Delta}(\omega)(\omega_1 \odot \omega_2))$ and $(\phi_1(\omega_1) \otimes \phi_2(\omega_2))\,\phi(\omega) = (\phi_1 \odot \phi_2)((\omega_1 \odot \omega_2)\hat{\Delta}(\omega))$ for every $\omega, \omega_1, \omega_2 \in \hat{A}$.
We have moreover that $\phi(\hat{A})(C_1 \otimes C_2)$ is dense in $C_1 \otimes C_2$.*

Proof : Define $B_1 = \phi_1(\hat{A})$ and $B_2 = \phi_2(\hat{A})$. Then B_1 is a sub*-algebra of $M(C_1)$ such that $B_1 C_1$ is dense in C_1 and $B_2 C_2$ is dense in C_2. This implies that $B_1 \odot B_2$ is a sub*-algebra of $M(C_1 \otimes C_2)$ such that $(B_1 \odot B_2)(C_1 \otimes C_2)$ is dense $C_1 \otimes C_2$.
Consider ϕ_k as an algebraically non-degenerate *-homomorphism from $\hat{A}$ into B_k $(k = 1,2)$ and define $\tilde{\phi} = (\phi_1 \odot \phi_2)\Delta$ in the algebraic way. So $\tilde{\phi}$ is an algebraically non-degenerate *-homomorphism from $\hat{A}$ into $M(B_1 \odot B_2)$.
By lemma 3.1, there exists a unique mapping ϕ from $\hat{A}$ into $M(C_1 \otimes C_2)$ such that $\phi(\omega)\, b = \tilde{\phi}(\omega)\, b$ and $b\,\phi(\omega) = b\,\tilde{\phi}(\omega)$ for every $b \in B_1 \odot B_2$. Because $\tilde{\phi}$ is a *-homomorphism, we get easily that ϕ is a *-homomorphism.
We have also for every $\omega_1, \omega_2, \omega \in \hat{A}$ that

$$\begin{aligned}
&\phi(\omega)\,(\phi_1(\omega_1) \otimes \phi_2(\omega_2)) = \tilde{\phi}(\omega)\,(\phi_1(\omega_1) \otimes \phi_2(\omega_2)) \\
&\quad = (\phi_1 \odot \phi_2)(\hat{\Delta}(\omega))\,(\phi_1(\omega_1) \otimes \phi_2(\omega_2)) = (\phi_1 \odot \phi_2)(\hat{\Delta}(\omega)(\omega_1 \odot \omega_2)) \ .
\end{aligned}$$

The other equality is proven in a similar way.
From this, we get easily that $\phi(\hat{A})(B_1 \odot B_2) = B_1 \odot B_2$ which implies that $\phi(\hat{A})(C_1 \otimes C_2)$ is dense in $C_1 \otimes C_2$. ∎

By duality, we get the following results :

Proposition 3.7 *Consider $a \in A$. Then there exists a positive number M such that $\|\phi(a)\| \leq M$ for every C*-algebra C and every *-homomorphism ϕ from A into C.*

Proposition 3.8 *Consider C*-algebras C_1, C_2 and *-homomorphisms ϕ_1 from A into $M(C_1)$ and ϕ_2 from A into $M(C_2)$ such that $\phi_1(A)C_1$ is dense in C_1 and $\phi_2(A)C_2$ is dense in C_2.*
Then there exists a unique *-homomorphism ϕ from A into $M(C_1 \otimes C_2)$ such that $(\phi_1(a_1) \otimes \phi_2(a_2))\,\phi(a) = (\phi_1 \odot \phi_2)((a_1 \otimes a_2)\Delta(a))$ and $\phi(a)\,(\phi_1(a_1) \otimes \phi_2(a_2)) = (\phi_1 \odot \phi_2)(\Delta(a)(a_1 \otimes a_2))$ for every $a \in A$ and $a_1, a_2 \in A$. We have moreover that $\phi(A)(C_1 \otimes C_2)$ is dense in $C_1 \otimes C_2$.

We are now in a position to define the universal C*-algebra associated to the *-algebra A together with a comultiplication on it.

Definition 3.9 *We define the norm $\|.\|_u$ on A such that*

$$\|a\|_u = \sup\{\, \|\phi(a)\| \mid C \text{ a C}^*\text{-algebra and } \phi \text{ a }^*\text{-homomorphism from } A \text{ into } C\,\} \ .$$

Then $\|.\|_u$ is a C*-norm on A, we define A_u to be the completion of A with respect to $\|.\|_u$, so A_u is a C*-algebra.

Remark 3.10 Proposition 3.7 implies that the norm is finite. The regular representation π_r guarantees that $\|.\|_u$ is a norm and not merely a semi-norm.

Notation 3.11 *We define π_u to be the identity mapping from A into A_u. Hence, π_u is an injective *-homomorphism from A into A_u such that $\pi_u(A)$ is dense in A_u.*

Restating the definition of the norm in terms of the mapping π_u, we have the following equality.

Proposition 3.12 *Consider $a \in A$. Then we have the following equality :*

$$\|\pi_u(a)\| = \sup \{ \|\phi(a)\| \mid C \text{ a } C^*\text{-algebra and } \phi \text{ a } {}^*\text{-homomorphism from } A \text{ into } C \} .$$

By the definition of the norm, we have the following universality property.

Proposition 3.13 *Consider a C^*-algebra C and a *-homomorphism ϕ from A into C. Then there exists a unique *-homomorphism θ from A_u into C such that $\theta \circ \pi_u = \phi$.*

If we apply proposition 3.8 with ϕ equal to π_u and combine this with the previous universality property, we get our comultiplication on A_u :

Definition 3.14 *There exists a unique non-degenerate *-homomorphism Δ_u from A_u into $M(A_u \otimes A_u)$ such that $(\pi_u \odot \pi_u)(x)\, \Delta_u(\pi_u(a)) = (\pi_u \odot \pi_u)(x\, \Delta(a))$ and $\Delta_u(\pi_u(a))\, (\pi_u \odot \pi_u)(x) = (\pi_u \odot \pi_u)(\Delta(a)\, x)$ for every $a \in A$ and $x \in A \odot A$.*

As in section 2 of [9], this definition implies for instance that $(\pi_u(a) \otimes 1)\Delta_u(\pi_u(b)) = (\pi_u \odot \pi_u)((a \otimes 1)\Delta(b))$ for every $a, b \in A$.

As before, this implies the next theorem.

Theorem 3.15 *The mapping Δ_u is a non-degenerate *-homomorphism from A_u into $M(A_u \otimes A_u)$ such that :*

1. $(\Delta_u \otimes \iota)\Delta_u = (\iota \otimes \Delta_u)\Delta_u$
2. *The vector spaces $\Delta_u(A_u)(A_u \otimes 1)$ and $\Delta_u(A_u)(1 \otimes A_u)$ are dense subspaces of $A_u \otimes A_u$.*

We call (A_u, Δ_u) the universal bi-C*-algebra associated to (A, Δ).

Thanks to the universality property of A_u (proposition 3.13), the counit on A can also be extended to A_u :

Definition 3.16 *We define ε_u as the unique *-homomorphism from A_u into $\mathbb{C}$ such that $\varepsilon_u \circ \pi_u = \varepsilon$.*

It is not difficult to extend the counit property from A to A_u :

Proposition 3.17 *We have for every $a \in A_u$ that $(\varepsilon_u \otimes \iota)\Delta_u(a) = (\iota \otimes \varepsilon_u)\Delta_u(a) = a$.*

This implies immediately that the mapping Δ_u is injective.

In the rest of this paper, we will need regularly to switch between A_u and A_r. The regular representation π_r together with the universality property (proposition 3.13) allows us to make the bridge :

Definition 3.18 *We define π to be the unique *-homomorphism from A_u into A_r such that $\pi \circ \pi_u = \pi_r$.*

Notice that π here is something different than in [9] !

By definition 3.14 and lemma 2.8 of [9], we have of course that $(\pi \otimes \pi)\Delta_u = \Delta_r\, \pi$.

4 The left regular corepresentation

In this section, we introduce the left regular corepresentation of (A_u, Δ_u). This corepresentation will be essential to transform objects connected with (A_r, Δ_r) to objects connected with (A_u, Δ_u). The first application of this principle can be found in the next section.

Remember that $A_u \otimes H$ is a Hilbert-C*-module in a natural way and let $\mathcal{L}(A_u \otimes H)$ denote the set of adjointable mappings from $A_u \otimes H$ into $A_u \otimes H$. Remember also that $\mathcal{L}(A_u \otimes H) = M(A_u \otimes B_0(H))$.

Definition 4.1 *There exists a unique unitary $V \in \mathcal{L}(A_u \otimes H)$ such that $V(\pi_u \odot \Lambda)(\Delta(b)(a \otimes 1)) = \pi_u(a) \otimes \Lambda(b)$ for every $a, b \in A$. We call V the left regular corepresentation of (A_u, Δ_u).*

This definition is possible because $\Delta(A)(A \otimes 1) = A \odot A$ and because of the left invariance of φ.

In the same way as in proposition 2.2 of [9], we get the following alternative formula for V :

Result 4.2 *We have for every $a, b \in A$ that $V(\pi_u(a) \otimes \Lambda(b)) = (\pi_u \odot \Lambda)\big((S^{-1} \odot \iota)(\Delta(b))\,(a \otimes 1)\big)$.*

Equation (1) of [9] has its obvious variant in the universal case :

Lemma 4.3 *We have for every $a, b \in A$ that $(\iota \odot \omega_{\Lambda(a),\Lambda(b)})(V) = \pi_u\big((\iota \odot \varphi)(\Delta(b^*)(1 \otimes a))\big)$.*

Proof : We have for every $c, d \in A$ that

$$\begin{aligned}
&\pi_u(d)^* \, (\iota \odot \omega_{\Lambda(a),\Lambda(b)})(V) \, \pi_u(c) = \langle V(\pi_u(c) \otimes \Lambda(a)), \pi_u(d) \otimes \Lambda(b) \rangle \\
&\quad = \langle \pi_u(c) \otimes \Lambda(a), V^*(\pi_u(d) \otimes \Lambda(b)) \rangle = \langle \pi_u(c) \otimes \Lambda(a), (\pi_u \odot \Lambda)(\Delta(b)(d \otimes 1)) \rangle \\
&\quad = (\pi_u \odot \varphi)((d^* \otimes 1)\Delta(b^*)(c \otimes a)) = \pi_u\big((\iota \odot \varphi)((d^* \otimes 1)\Delta(b^*)(c \otimes a))\big) \\
&\quad = \pi_u(d^*(\iota \odot \varphi)(\Delta(b^*)(1 \otimes a))\, c) = \pi_u(d)^* \, \pi_u\big((\iota \odot \varphi)(\Delta(b^*)(1 \otimes a))\big) \, \pi_u(c) \ .
\end{aligned}$$

This implies that $(\iota \odot \omega_{\Lambda(a),\Lambda(b)})(V) = \pi_u\big((\iota \odot \varphi)(\Delta(b^*)(1 \otimes a))\big)$. ■

As a consequence, we find the following result.

Corollary 4.4 *The set $\{\, (\iota \odot \omega)(V) \mid \omega \in B_0(H)^* \,\}$ is a dense subspace of A_u.*

Lifting V to H gives W :

Result 4.5 *We have that $(\pi \otimes \iota)(V) = W$.*

Proof : Choose $a, b \in A$. Using lemma 4.3 and equation (1) of [9], we get that

$$\begin{aligned}
&(\iota \otimes \omega_{\Lambda(a),\Lambda(b)})((\pi \otimes \iota)(V)) = \pi((\iota \otimes \omega_{\Lambda(a),\Lambda(b)})(V)) = \pi\big(\, \pi_u(\, (\iota \odot \varphi)(\Delta(b^*)(1 \otimes a))\,)\,\big) \\
&\quad = \pi_r\big((\iota \odot \varphi)(\,\Delta(b^*)(1 \otimes a))\big) = (\iota \otimes \omega_{\Lambda(a),\Lambda(b)})(W) \ .
\end{aligned}$$

Consequently, $(\pi \otimes \iota)(V) = W$. ■

Notation 4.6 *For every $p, q \in H$, we denote $\omega'_{p,q} \in A_u^*$ such that $\omega'_{p,q}(x) = \langle \pi(x)p, q \rangle$ for every $x \in A_u$.*

Because $(\pi \otimes \iota)(V) = W$, lemma 2.20 of [9] implies the following equality :

Result 4.7 *We have for every $a, b \in A$ that $(\omega'_{\Lambda(a),\Lambda(b)} \otimes \iota)(V) = \hat{\pi}_r(a\varphi b^*)$.*

Corollary 4.8 *We have that $\hat{A}_r$ is a subset of the closure of $\{ (\omega \otimes \iota)(V) \mid \omega \in A_u^* \}$ in $B_0(H)$.*

The left regular corepresentation lifts the algebraic universal corepresentation X from $A \odot \hat{A}$ to $A_u \otimes \hat{A}_r$:

Proposition 4.9 *We have for every $a, b \in A$ and $x \in A \odot \hat{A}$ that $V(\pi_u \odot \hat{\pi}_r)(x) = (\pi_u \odot \hat{\pi}_r)(X\,x)$ and $(\pi_u \odot \hat{\pi}_r)(x)\,V = (\pi_u \odot \hat{\pi}_r)(x\,X)$*

The proof of the first equality is completely the same as the proof of lemma 2.21 of [9]. The second one follows from the first.

Corollary 4.10 *We have the equalities $V(\pi_u(A) \odot \hat{\pi}_r(\hat{A})) = \pi_u(A) \odot \hat{\pi}_r(\hat{A})$ and $(\pi_u(A) \odot \hat{\pi}_r(\hat{A}))\,V = \pi_u(A) \odot \hat{\pi}_r(\hat{A})$.*

Corollary 4.11 *The element V belongs to $M(A_u \otimes \hat{A}_r)$.*

Because $(\Delta \odot \iota)(X) = X_{13}\,X_{23}$ and $(\iota \odot \hat{\Delta})(X) = X_{12}\,X_{13}$, proposition 4.9 implies the following result :

Proposition 4.12 *We have that $(\Delta_u \otimes \iota)(V) = V_{13}\,V_{23}$ and $(\iota \odot \hat{\Delta}_r)(V) = V_{12}\,V_{23}$.*

So V is a corepresentation of (A_u, Δ_u) on A_r and a corepresentation of (A_r, Δ_r) on A_u.

As usual, V implements the comultiplication :

Proposition 4.13 *We have for every $a \in A_u$ that $(\iota \otimes \pi)\Delta_u(a) = V^*(1 \otimes \pi(a))V$.*

Proof : Choose $x \in A$. Take $b, c \in A$. Then there exists $e \in A$ such that $(e\otimes 1)\Delta(c)(b\otimes 1) = \Delta(c)(b\otimes 1)$. This implies that

$$\begin{aligned}
&V^*(1 \otimes \pi(\pi_u(x)))\,(\pi_u(b) \otimes \Lambda(c)) = V^*(1 \otimes \pi_r(x))(\pi_u(b) \otimes \Lambda(c)) \\
&\quad = V^*(\pi_u(b) \otimes \Lambda(xc)) = (\pi_u \odot \Lambda)(\Delta(xc)(b \otimes 1)) \\
&\quad = (\pi_u \odot \Lambda)(\Delta(x)\Delta(c)(b \otimes 1)) = (\pi_u \odot \Lambda)(\Delta(x)(e \otimes 1)\Delta(c)(b \otimes 1)) \\
&\quad = (\pi_u \odot \pi_r)(\Delta(x)(e \otimes 1))\,(\pi_u \odot \Lambda)(\Delta(c)(b \otimes 1)) \\
&\quad = (\iota \otimes \pi)\big((\pi_u \odot \pi_u)(\Delta(x)(e \otimes 1))\big)\,(\pi_u \odot \Lambda)(\Delta(c)(b \otimes 1)) \\
&\quad = (\iota \otimes \pi)(\Delta_u(\pi_u(x))(\pi_u(e) \otimes 1))\,(\pi_u \odot \Lambda)(\Delta(c)(b \otimes 1)) \\
&\quad = (\iota \otimes \pi)\Delta_u(\pi_u(x))\,(\pi_u(e) \otimes 1)\,(\pi_u \odot \Lambda)(\Delta(c)(b \otimes 1)) \\
&\quad = (\iota \otimes \pi)\Delta_u(\pi_u(x))\,(\pi_u \odot \Lambda)(\Delta(c)(b \otimes 1)) = (\iota \otimes \pi)\Delta_u(\pi_u(x))\,V^*(\pi_u(b) \otimes \Lambda(c))\ .
\end{aligned}$$

So we see that $V^*(1 \otimes \pi(\pi_u(x))) = (\iota \otimes \pi)\Delta_u(\pi_u(x))\,V^*$. Because $\pi_u(A)$ is dense in A_u, we get that $V^*(1 \otimes \pi(a)) = (\iota \otimes \pi)(\Delta_u(a))V^*$. ■

In the rest of this section, we prove a formula which formally says that $(\iota \otimes \hat{S})(V) = V^*$. It will prove useful in the next section. First, we need a lemma.

Lemma 4.14 *Consider $a, b \in A$ and $\omega \in A_u^*$. Then*

$$(\pi_u(a)\omega\pi_u(b) \otimes \iota)(V^*)\,\Lambda(x) = \Lambda\big((\omega\circ\pi_u \odot \iota)((b \otimes 1)\Delta(x)(a \otimes 1))\big)$$

for every $x \in A$.

Proof : We have for every $y \in A$ that

$$\begin{aligned}
&\langle(\pi_u(a)\omega\pi_u(b) \otimes \iota)(V^*)\,\Lambda(x), \Lambda(y)\rangle = \omega(\langle V^*(\pi_u(a) \otimes \Lambda(x)), \pi_u(b^*) \otimes \Lambda(y)\rangle) \\
&\quad = \omega(\langle(\pi_u \odot \Lambda)(\Delta(x)(a \otimes 1)), \pi_u(b^*) \otimes \Lambda(y)\rangle) = \omega\big((\pi_u \odot \varphi)((b \otimes y^*)\Delta(x)(a \otimes 1))\big) \\
&\quad = \varphi\big((\omega\circ\pi_u \odot \iota)((b \otimes y^*)\Delta(x)(a \otimes 1))\big) = \varphi\big(y^*(\omega\circ\pi_u \odot \iota)((b \otimes 1)\Delta(x)(a \otimes 1))\big) \\
&\quad = \langle\Lambda\big((\omega\circ\pi_u \odot \iota)((b \otimes 1)\Delta(x)(a \otimes 1))\big), \Lambda(y)\rangle \ .
\end{aligned}$$

This implies that $(\pi_u(a)\omega\pi_u(b) \otimes \iota)(V^*)\,\Lambda(x) = \Lambda\big((\omega\circ\pi_u \odot \iota)((b \otimes 1)\Delta(x)(a \otimes 1))\big)$. ■

Proposition 4.15 *Consider $p \in D(\nabla^{\frac{1}{2}})$ and $q \in D(\nabla^{-\frac{1}{2}})$. Then $(\iota\otimes\omega_{J\nabla^{\frac{1}{2}}p, J\nabla^{-\frac{1}{2}}q})(V^*) = (\iota\otimes\omega_{q,p})(V)$.*

Proof : Choose $a, b \in A$ and $\omega \in A_u^*$. Take $x \in A$. We know from the remarks after proposition 2.15 that $\Lambda(x)$ belongs to $D(\nabla^{\frac{1}{2}})$ and $J\nabla^{\frac{1}{2}}\Lambda(x) = \Lambda(x^*)$. Therefore

$$\begin{aligned}
&(\pi_u(a)\omega\pi_u(b^*))\big((\iota \otimes \omega_{J\nabla^{\frac{1}{2}}\Lambda(x), J\nabla^{-\frac{1}{2}}q})(V^*)\big) = (\pi_u(a)\omega\pi_u(b^*))\big((\iota \otimes \omega_{\Lambda(x^*), J\nabla^{-\frac{1}{2}}q})(V^*)\big) \\
&\quad = \langle(\pi_u(a)\omega\pi_u(b^*) \otimes \iota)(V^*)\,\Lambda(x^*), J\nabla^{-\frac{1}{2}}q\rangle = \langle\Lambda\big((\omega\circ\pi_u \odot \iota)((b^* \otimes 1)\Delta(x^*)(a \otimes 1))\big), J\nabla^{-\frac{1}{2}}q\rangle \qquad (*)
\end{aligned}$$

where we used the previous lemma in the last equality. By the remarks after proposition 2.15, we know that also $\Lambda\big((\overline{\omega}\circ\pi_u \odot \iota)((a^* \otimes 1)\Delta(x)(b \otimes 1))\big)$ belongs to $D(\nabla^{\frac{1}{2}})$ and

$$J\nabla^{\frac{1}{2}}\Lambda\big((\overline{\omega}\circ\pi_u \odot \iota)((a^* \otimes 1)\Delta(x)(b \otimes 1))\big) = \Lambda\big((\omega\circ\pi_u \odot \iota)((b^* \otimes 1)\Delta(x^*)(a \otimes 1))\big) \ .$$

Substituting this in equality (*), we get that

$$(\pi_u(a)\omega\pi_u(b^*))\big((\iota \otimes \omega_{J\nabla^{\frac{1}{2}}\Lambda(x), J\nabla^{-\frac{1}{2}}q})(V^*)\big) = \langle J\nabla^{\frac{1}{2}}\Lambda\big((\overline{\omega}\circ\pi_u \odot \iota)((a^* \otimes 1)\Delta(x)(b \otimes 1))\big), J\nabla^{-\frac{1}{2}}q\rangle \ .$$

This implies that

$$\begin{aligned}
&(\pi_u(a)\omega\pi_u(b^*))\big((\iota \otimes \omega_{J\nabla^{\frac{1}{2}}\Lambda(x), J\nabla^{-\frac{1}{2}}q})(V^*)\big) \\
&\quad = \langle q, \Lambda\big((\overline{\omega}\circ\pi_u \odot \iota)((a^* \otimes 1)\Delta(x)(b \otimes 1))\big)\rangle = \langle q, (\pi_u(b)\overline{\omega}\,\pi_u(a^*) \otimes \iota)(V^*)\Lambda(x)\rangle \\
&\quad = \langle(\pi_u(a)\omega\pi_u(b^*) \otimes \iota)(V)\,q, \Lambda(x)\rangle = (\pi_u(a)\omega\pi_u(b^*))((\iota \otimes \omega_{q,\Lambda(x)})(V)) \ .
\end{aligned}$$

From this all, we conclude that

$$(\iota \otimes \omega_{J\nabla^{\frac{1}{2}}\Lambda(x), J\nabla^{-\frac{1}{2}}q})(V^*) = (\iota \otimes \omega_{q,\Lambda(x)})(V) \ .$$

Because $\Lambda(A)$ is a core for $\nabla^{\frac{1}{2}}$ (see again the remarks after proposition 2.15), the proposition follows. ■

Corollary 4.16 *Consider $p \in D(\nabla^{\frac{1}{2}})$ and $q \in D(\nabla^{-\frac{1}{2}})$. Then $(\iota \otimes \omega_{p,q})(V^*) = (\iota \otimes \omega_{J\nabla^{-\frac{1}{2}}q, J\nabla^{\frac{1}{2}}p})(V)$.*

5 Lifting bi-C*-isomorphisms from the reduced to the universal C*-algebra

In [9], we saw that the reduced quantum group (A_r, Δ_r) possesses a very rich structure : a left and right Haar weight which are KMS, a polar decomposition of the antipode, ...
We will show that this rich structure on (A_r, Δ_r) can be transported to (A_u, Δ_u). In this section, we will provide the most important method to do so.

For the rest of this section, we fix *-automorphisms α, β on A_r such that $(\alpha \otimes \beta)\Delta_r = \Delta_r\, \alpha$.

Lemma 5.1 *We have that* $(\beta \otimes \beta)\Delta_r = \Delta_r\, \beta$.

Proof : We have that

$$\begin{aligned}(\alpha \otimes \Delta_r\, \beta)\Delta_r &= (\iota \otimes \Delta_r)\Delta_r\, \alpha = (\Delta_r \otimes \iota)\Delta_r\, \alpha \\ &= (\Delta_r \otimes \iota)(\alpha \otimes \beta)\Delta_r = (\alpha \otimes \beta \otimes \beta)(\Delta_r \otimes \iota)\Delta_r = (\alpha \otimes (\beta \otimes \beta)\Delta_r)\Delta_r\end{aligned}$$

which implies that $\Delta_r\, \beta = (\beta \otimes \beta)\Delta_r$. ■

Hence, using proposition 7.1 of [9], we get the following result.

Corollary 5.2 *There exists a unique strictly positive number r such that $\varphi_r\, \beta = r\, \varphi_r$.*

This relative invariance can be extended to α :

Proposition 5.3 *We have also that $\varphi_r\, \alpha = r\, \varphi_r$.*

Proof : Choose $a \in \mathcal{M}_{\varphi_r}^+$.
Take $\omega \in (A_r)^*_+$. Then

$$(\omega \otimes \iota)\Delta_r(\alpha(a)) = \beta((\omega\alpha \otimes \iota)\Delta_r(a)) \ . \qquad (*)$$

Because a belongs to $\mathcal{M}_{\varphi_r}^+$ and φ_r is left invariant, we get that $(\omega\alpha \otimes \iota)\Delta_r(a)$ belongs to $\mathcal{M}_{\varphi_r}^+$ and

$$\varphi_r((\omega\alpha \otimes \iota)\Delta_r(a)) = \omega(\alpha(1))\, \varphi_r(a) = \omega(1)\, \varphi_r(a) \ .$$

Because $\varphi_r\, \beta = r\, \varphi_r$, this implies that $\beta((\omega\alpha \otimes \iota)\Delta_r(a))$ belongs to $\mathcal{M}_{\varphi_r}^+$ and

$$\varphi_r\big(\beta((\omega\alpha \otimes \iota)\Delta_r(a))\big) = r\, \varphi_r((\omega\alpha \otimes \iota)\Delta_r(a)) = r\, \omega(1)\, \varphi_r(a) \ .$$

Looking at equality (*), this implies that $(\omega \otimes \iota)\Delta_r(\alpha(a))$ belongs to $\mathcal{M}_{\varphi_r}^+$ and

$$\varphi_r\big((\omega \otimes \iota)\Delta_r(\alpha(a))\big) = r\, \omega(1)\, \varphi_r(a) \ .$$

By theorem of 3.11 of [12], we get that $\alpha(a)$ belongs to $\mathcal{M}_{\varphi_r}^+$.

Using the left invariance of φ_r, we get moreover for every $\theta \in (A_r)^*_+$ that

$$\theta(1)\, \varphi_r(\alpha(a)) = \varphi_r\big((\theta \otimes \iota)\Delta(\alpha(a))\big) = r\, \theta(1)\, \varphi_r(a)$$

which implies that $\varphi_r(\alpha(a)) = r\, \varphi_r(a)$. ■

Notation 5.4 *We define the unitary elements $u, v \in B(H)$ such that $u\Lambda_r(a) = r^{-\frac{1}{2}}\Lambda_r(\alpha(a))$ and $v\Lambda_r(a) = r^{-\frac{1}{2}}\Lambda_r(\beta(a))$ for every $a \in \mathcal{N}_{\varphi_r}$.*

Then we have the following results :

Result 5.5 *We have for every $x \in A_r$ that $\alpha(x) = uxu^*$ and $\beta(x) = vxv^*$.*

Result 5.6 *We have the following commutation relations :*

1. *$Ju = uJ$ and $\nabla u = u\nabla$*
2. *$Jv = vJ$ and $\nabla v = v\nabla$*

The equality $(\alpha \otimes \beta)\Delta_r = \Delta_r\,\alpha$ implies the following commutation relations between u, v and W.

Proposition 5.7 *The following commutation relations hold :*

1. *$W(u \otimes v) = (u \otimes u)W$*
2. *$W(v \otimes v) = (v \otimes v)W$*

Proof :

1. Choose $a, b \in \mathcal{N}_{\varphi_r}$.

   Then $\alpha(a), \alpha(b)$ belong to $\mathcal{N}_{\varphi_r}$ and $u\Lambda_r(a) = r^{-\frac{1}{2}}\,\Lambda_r(\alpha(a))$ and $u\Lambda_r(b) = r^{-\frac{1}{2}}\,\Lambda_r(\alpha(b))$.

   Because $\alpha(a), \alpha(b)$ belong to $\mathcal{N}_{\varphi_r}$, proposition 2.17 implies that $\Delta(\alpha(b))(\alpha(a)\otimes 1)$ belongs to $\mathcal{N}_{\varphi_r\otimes\varphi_r}$ and

   $$(\Lambda_r\otimes\Lambda_r)(\Delta(\alpha(b))(\alpha(a)\otimes 1)) = W^*(\Lambda_r(\alpha(a))\otimes\Lambda_r(\alpha(b))) = r\;(W^*(u\otimes u))(\Lambda_r(a)\otimes\Lambda_r(b))\ . \qquad \text{(a)}$$

   Because a, b belong to $\mathcal{N}_{\varphi_r}$, proposition 2.17 implies that also $\Delta(b)(a \otimes 1)$ belongs to $\mathcal{N}_{\varphi_r\otimes\varphi_r}$ and

   $$(\Lambda_r \otimes \Lambda_r)(\Delta(b)(a \otimes 1)) = W^*(\Lambda_r(a) \otimes \Lambda_r(b))\ .$$

   This implies that $(\alpha \otimes \beta)(\Delta(b)(a \otimes 1))$ belongs to $\mathcal{N}_{\varphi_r\otimes\varphi_r}$ and

   $$\begin{aligned}(\Lambda_r \otimes \Lambda_r)\big((\alpha \otimes \beta)(\Delta(b)(a \otimes 1))\big) &= r\;(u \otimes v)(\Lambda_r \otimes \Lambda_r)(\Delta(b)(a \otimes 1)) \\ &= r\;((u \otimes v)W^*)(\Lambda_r(a) \otimes \Lambda_r(b))\ . \qquad \text{(b)}\end{aligned}$$

   Because $(\alpha \otimes \beta)(\Delta(b)(a \otimes 1)) = \Delta(\alpha(b))(\alpha(a) \otimes 1)$, equalities (a) and (b) imply that

   $$(W^*(u \otimes u))(\Lambda_r(a) \otimes \Lambda_r(b)) = ((u \otimes v)W^*)(\Lambda_r(a) \otimes \Lambda_r(b))\ .$$

   So we get that $W^*(u \otimes u) = (u \otimes v)W^*$.

2. The second equality is proven in the same way as the first one.

■

Corollary 5.8 *We have for every $\omega \in B_0(H)^*$ that $\alpha((\iota \otimes \omega)(W)) = (\iota \otimes v\omega u^*)(W)$ and $\beta((\iota \otimes \omega)(W)) = (\iota \otimes v\omega v^*)(W)$.*

Using these commutation relations, we get the following result.

Result 5.9 *The following equalities hold :*

1. $\hat{A}_r = u^* \hat{A}_r v = v^* \hat{A}_r u$
2. $\hat{A}_r = u^* \hat{A}_r u = v^* \hat{A}_r v$

Proof : By proposition 5.7, we have for every $\omega \in B_0(H)^*$ that

$$u^*(\omega \otimes \iota)(W)v = (\omega \otimes \iota)((1 \otimes u^*)W(1 \otimes v)) = (\omega \otimes \iota)((u \otimes 1)W(u^* \otimes 1)) = (u^*\omega u \otimes \iota)(W) \ .$$

Because $\hat{A}_r$ is the closure of $\{ (\omega \otimes \iota)(W) \mid \omega \in B_0(H)^* \}$, this equality implies that $u^* \hat{A}_r v = \hat{A}_r$.
Taking the adjoint of this equation gives us that $v^* \hat{A}_r u = \hat{A}_r$.
We have moreover that

$$u^* (\hat{A}_r)^2 u = (u^* \hat{A}_r v)(v^* \hat{A}_r u) = (\hat{A}_r)^2 \ .$$

Because $(\hat{A}_r)^2$ is a dense subset of $\hat{A}_r$, this implies that $u^* \hat{A}_r u = \hat{A}_r$. In a similar way, we get that $v^* \hat{A}_r v = \hat{A}_r$. ■

Using this last result, it is not difficult to infer the following one.

Result 5.10 *The following equalities hold :*

1. $M(\hat{A}_r) = u^* M(\hat{A}_r) v = v^* M(\hat{A}_r) u$
2. $M(\hat{A}_r) = u^* M(\hat{A}_r) u = v^* M(\hat{A}_r) v$

This last result will imply the following lemma.

Lemma 5.11 *Consider $\omega, \theta \in B_0(H)^*$ such that $(\iota \otimes \omega)(V) = (\iota \otimes \theta)(V)$. Then $(\iota \otimes v\omega u^*)(V) = (\iota \otimes v\theta u^*)(V)$ and $(\iota \otimes v\omega v^*)(V) = (\iota \otimes v\theta v^*)$.*

Proof : We have for every $\eta \in A_u^*$ that

$$\omega((\eta \otimes \iota)(V)) = \eta((\iota \otimes \omega)(V)) = \eta((\iota \otimes \theta)(V)) = \theta((\eta \otimes \iota)(V)) \ .$$

Therefore, corollary 4.8 implies that $\omega(x) = \theta(x)$ for every $x \in \hat{A}_r$. Therefore, $\omega(x) = \theta(x)$ for every $x \in M(\hat{A}_r)$.

Take $\eta \in A_u^*$. Because V belongs to $M(A_u \otimes \hat{A}_r)$, we get that $(\eta \otimes \iota)(V)$ belongs to $M(\hat{A}_r)$. Therefore, the previous result implies that $u^*(\eta \otimes \iota)(V)v$ belongs to $M(\hat{A}_r)$.
By the first part of the proof, we know that

$$\omega(u^*(\eta \otimes \iota)(V)v) = \theta(u^*(\eta \otimes \iota)(V)v) \ ,$$

implying that $\eta((\iota \otimes v\omega u^*)(V)) = \eta((\iota \otimes v\theta u^*)(V))$.
Consequently, $(\iota \otimes v\omega u^*)(V) = (\iota \otimes v\theta u^*)(V)$. Similarly, $(\iota \otimes v\omega v^*)(V) = (\iota \otimes v\theta v^*)(V)$. ■

This allows us to prove the following proposition :

Proposition 5.12 *There exists a unique linear mapping F from $\pi_u(A)$ into A_u such that $F((\iota \otimes \omega_{p,q})(V)) = (\iota \otimes \omega_{vp,uq})(V)$ for every $p, q \in \Lambda(A)$.*

Proof : By the previous lemma, we have for every $p_1, \ldots, p_m, q_1, \ldots, q_m$, $r_1, \ldots, r_n, s_1, \ldots, s_n \in \Lambda(A)$ such that

$$\sum_{i=1}^{m} (\iota \otimes \omega_{p_i,q_i})(V) = \sum_{j=1}^{n} (\iota \otimes \omega_{r_j,s_j})(V)$$

that

$$\sum_{i=1}^{m} (\iota \otimes \omega_{vp_i,uq_i})(V) = \sum_{j=1}^{n} (\iota \otimes \omega_{vr_j,us_j})(V) \ .$$

This implies that we can define a mapping F from $\pi_u(A)$ into A_u such that

$$F\big(\sum_{i=1}^{m} (\iota \otimes \omega_{p_i,q_i})(V) \big) = \sum_{i=1}^{m} (\iota \otimes \omega_{vp_i,uq_i})(V)$$

for every $m \in \mathbb{N}$ and $p_1, \ldots, p_m, q_1, \ldots, q_m \in \Lambda(A)$. It is easy to check that F is linear. ■

Result 5.13 *The mapping F is multiplicative.*

Proof : Choose $p, q, r, s \in \Lambda(A)$.
Define $\omega \in B_0(H)^*$ such that $\omega(x) = \langle W(x \otimes 1)W^*(p \otimes r), q \otimes s \rangle$ for every $x \in B_0(H)$. Using proposition 4.12, we get that

$$\begin{aligned}(\iota \otimes \omega_{p,q})(V)\,(\iota \otimes \omega_{r,s})(V) &= (\iota \otimes \omega_{p,q} \otimes \omega_{r,s})(V_{12}V_{13}) \\ &= (\iota \otimes \omega_{p,q} \otimes \omega_{r,s})(W_{23}V_{12}W_{23}^*) = (\iota \otimes \omega)(V) \ .\end{aligned}$$

Therefore, lemma 5.11 implies that

$$F\big((\iota \otimes \omega_{p,q})(V)\,(\iota \otimes \omega_{r,s})(V) \big) = (\iota \otimes v\omega u^*)(V) \ . \qquad (*)$$

Now we have for every $x \in B_0(H)$ that

$$\begin{aligned}(v\omega u^*)(x) &= \omega(u^* x v) = \langle W(u^* x v \otimes 1)W^*(p \otimes r), q \otimes s \rangle \\ &= \langle W(u^* \otimes v^*)(x \otimes 1)(v \otimes v)W^*(p \otimes r), q \otimes s \rangle = \langle W(x \otimes 1)W^*(vp \otimes vr), uq \otimes us \rangle\end{aligned}$$

where we used proposition 5.7 in the last equality. Combining this result with equality (*), we see that

$$\begin{aligned}F\big((\iota \otimes \omega_{p,q})(V)\,(\iota \otimes \omega_{r,s})(V) \big) &= (\iota \otimes \omega_{vp,uq} \otimes \omega_{vr,us})(W_{23}V_{12}W_{23}^*) = (\iota \otimes \omega_{vp,uq} \otimes \omega_{vr,us})(V_{12}V_{13}) \\ &= (\iota \otimes \omega_{vp,uq})(V)\,(\iota \otimes \omega_{vr,us})(V) = F((\iota \otimes \omega_{p,q})(V))\,F((\iota \otimes \omega_{r,s})(V)) \ .\end{aligned}$$

The result follows by linearity. ■

Result 5.14 *The mapping F is selfadjoint.*

Proof : Choose $p, q \in \Lambda(A)$. We know from the remarks after proposition 2.15 that p belongs to $D(\nabla^{-\frac{1}{2}})$ and that q belongs to $D(\nabla^{\frac{1}{2}})$. Therefore, corollary 4.16 implies that

$$(\iota \otimes \omega_{p,q})(V)^* = (\iota \otimes \omega_{q,p})(V^*) = (\iota \otimes \omega_{J\nabla^{-\frac{1}{2}}p,\, J\nabla^{\frac{1}{2}}q})(V) \ .$$

Using lemma 5.11, this implies that

$$F((\iota \otimes \omega_{p,q})(V)^*) = (\iota \otimes \omega_{vJ\nabla^{-\frac{1}{2}}p,\, uJ\nabla^{\frac{1}{2}}q})(V) \ .$$

Result 5.6 implies that vp belongs to $D(\nabla^{-\frac{1}{2}})$, that uq belongs to $D(\nabla^{\frac{1}{2}})$ and $J\nabla^{-\frac{1}{2}}vp = vJ\nabla^{-\frac{1}{2}}p$ and $J\nabla^{\frac{1}{2}}uq = uJ\nabla^{\frac{1}{2}}q$. Hence,

$$\begin{aligned} F((\iota\otimes\omega_{p,q})(V)^*) &= (\iota\otimes\omega_{J\nabla^{-\frac{1}{2}}vp,J\nabla^{\frac{1}{2}}uq})(V) \\ &\overset{(*)}{=} (\iota\otimes\omega_{uq,vp})(V^*) = (\iota\otimes\omega_{vp,uq})(V)^* = F((\iota\otimes\omega_{p,q})(V))^* \end{aligned}$$

where corollary 4.16 was used once again in equality (*). The result follows by linearity. ■

Consequently, we have proven that F is a *-homomorphism from $\pi_u(A)$ into A_u. By proposition 3.13, this justifies the following definition :

Definition 5.15 *There exists a unique *-homomorphism α_u from A_u into A_u such that $\alpha_u((\iota\otimes\omega_{p,q})(V))$ $= (\iota\otimes\omega_{vp,uq})(V)$ for every $p,q\in\Lambda(A)$.*

Then we get immediately the following proposition.

Result 5.16 *The equality $(\alpha_u\otimes\iota)(V) = (1\otimes u^*)V(1\otimes v)$ holds.*

Corollary 5.17 *We have for every $\omega\in B_0(H)^*$ that $\alpha_u((\iota\otimes\omega)(V)) = (\iota\otimes v\omega u^*)(V)$.*

Proposition 5.18 *The mapping α_u is an *-automorphism on A_u.*

Proof : We have of course also the equality $(\alpha^{-1}\otimes\beta^{-1})\Delta = \Delta\alpha^{-1}$. Therefore we can do the same thing for α^{-1} as we did for α. So we get the existence of a * homomorphism θ from A_u into A_u such that $\theta((\iota\otimes\omega)(V)) = (\iota\otimes v^*\omega u)(V)$ for every $\omega\in B_0(H)$. Then we get immediately that

$$\alpha_u(\theta((\iota\otimes\omega)(V))) = \theta(\alpha_u((\iota\otimes\omega)(V))) = (\iota\otimes\omega)(V)$$

for every $\omega\in B_0(H)^*$.
From this, it follows that $\alpha_u\circ\theta = \theta\circ\alpha_u = \iota$. The proposition follows. ■

In a similar way, we can do the same things for β. So we get the following result.

Proposition 5.19 *There exists a unique *-automorphism β_u on A_u such that $\beta_u((\iota\otimes\omega)(V))$ $= (\iota\otimes v\omega v^*)(V)$ for every $\omega\in B_0(H)^*$.*

Result 5.20 *The equality $(\beta_u\otimes\iota)(V) = (1\otimes v^*)V(1\otimes v)$ holds.*

Furthermore, the commutation relation between α, β and Δ_r is transferred to the same commutation relation between α_u, β_u and Δ_u :

Proposition 5.21 *We have that $(\alpha_u\otimes\beta_u)\Delta_u = \Delta_u\,\alpha_u$.*

Proof : Using proposition 4.12, we have that

$$((\alpha_u \otimes \beta_u)\Delta_u \otimes \iota)(V) = (\alpha_u \otimes \beta_u \otimes \iota)(V_{13}V_{23}) = (\alpha_u \otimes \iota)(V)_{13}\,(\beta_u \otimes \iota)(V)_{23}\ .$$

Therefore, result 5.16 and 5.20 imply that

$$\begin{aligned}((\alpha_u \otimes \beta_u)\Delta_u \otimes \iota)(V) &= [(1 \otimes u^*)V(1 \otimes v)]_{13}\,[(1 \otimes v^*)V(1 \otimes v)]_{23} \\ &= (1 \otimes 1 \otimes u^*)V_{13}(1 \otimes 1 \otimes v)\,(1 \otimes 1 \otimes v^*)V_{23}(1 \otimes 1 \otimes v) = (1 \otimes 1 \otimes u^*)V_{13}V_{23}(1 \otimes 1 \otimes v)\ .\end{aligned}$$

Hence, using proposition 4.12 and result 5.16 once more, we get that

$$\begin{aligned}((\alpha_u \otimes \beta_u)\Delta_u \otimes \iota)(V) &= (1 \otimes 1 \otimes u^*)(\Delta_u \otimes \iota)(V)(1 \otimes 1 \otimes v) \\ &= (\Delta_u \otimes \iota)((1 \otimes u^*)V(1 \otimes v)) = (\Delta_u\,\alpha_u \otimes \iota)(V)\ .\end{aligned}$$

This last equality implies for every $\omega \in B_0(H)^*$ that

$$(\alpha_u \otimes \beta_u)\big(\Delta_u((\iota \otimes \omega)(V))\big) = \Delta_u\big(\alpha_u((\iota \otimes \omega)(V))\big)\ .$$

From this all, we can conclude that $(\alpha_u \otimes \beta_u)\Delta_u = \Delta_u\,\alpha_u$. ■

Because $(\pi \otimes \iota)(V) = W$, corollaries 5.8, 5.17 and proposition 5.19 imply easily the following result.

Result 5.22 *We have that $\pi\alpha_u = \alpha$ and $\pi\beta_u = \beta$.*

Remark 5.23 Later (proposition 11.8), we will prove that α_u and β_u are uniquely determined by the properties $\pi\alpha_u = \alpha$ and $(\alpha_u \otimes \beta_u)\Delta = \Delta\alpha_u$.

In a later section, we will also transform corepresentations from (A_r, Δ_r) to corepresentations of (A_u, Δ_u).

6 The algebraic nature of a unitary corepresentation of (A_r, Δ_r)

We will prove in this section that every unitary corepresentation of (A_r, Δ_r) is of an algebraic nature.

For the rest of this section, we fix a C*-algebra C and a unitary corepresentation $\mathcal{U}$ of (A_r, Δ_r) on C. We want to prove a generalization of proposition 7.7 of [9]. In order to do so, we need to single out a special sub-*-algebra of C.

Notation 6.1 *Define the set $B = \langle\, (\omega_{p,q} \otimes \iota)(\mathcal{U}) \mid p, q \in \Lambda(A)\,\rangle$, so B is a subspace of $M(C)$.*

For every $\omega \in \hat{A}$, there exists a unique element $\theta \in A_r^*$ such that $\theta \circ \pi_r = \omega$ and we define $\tilde{\omega} = \theta$. In this notation , we have for every $a, b \in A$ that $(a\varphi b^*)\tilde{} = \omega_{\Lambda(a),\Lambda(b)}$.
So we get that $B = \{\, (\tilde{\omega} \otimes \iota)(\mathcal{U}) \mid \omega \in \hat{A}\,\}$.

It is not so difficult to check that $(\omega_1\omega_2)\tilde{} = (\tilde{\omega}_1 \otimes \tilde{\omega}_2)\Delta_r$ for every $\omega_1, \omega_2 \in \hat{A}$.
Because $(\Delta_r \otimes \iota)(\mathcal{U}) = \mathcal{U}_{13}\mathcal{U}_{23}$, this implies easily that the mapping $\hat{A} \to M(C) : \omega \mapsto (\tilde{\omega} \otimes \iota)(\mathcal{U})$ is an homomorphism of algebras.

Therefore, we get the following result :

Result 6.2 *We have that $B^2 = B$.*

This algebra B is non-degenerate with respect to C :

Result 6.3 *We have that BC and CB are dense in C.*

Proof : Choose $c \in C$. There exists $a, b \in A$ and $x \in A_r$ such that $\langle x\,\Lambda(a), \Lambda(b)\rangle = 1$. We have for every $d \in A$ and $y \in C$ that

$$(\omega_{\Lambda(a),\Lambda(b)} \otimes \iota)(\mathcal{U}(\pi_r(d) \otimes y)) = (\omega_{\pi_r(d)\Lambda(a),\Lambda(b)} \otimes \iota)(\mathcal{U})\, y = (\omega_{\Lambda(da),\Lambda(b)} \otimes \iota)(\mathcal{U})\, y \in BC \ .$$

Because $\pi_r(A) \odot C$ is dense in $A_r \otimes C$, this implies for every $z \in A_r \otimes C$ that $(\omega_{\Lambda(a),\Lambda(b)} \otimes \iota)(\mathcal{U} z)$ belongs to $\overline{BC}$. Because $\mathcal{U}$ is unitary, this implies that $(\omega_{\Lambda(a),\Lambda(b)} \otimes \iota)(z)$ belongs to $\overline{BC}$ for every $z \in A_r \otimes C$. Because $c = (\omega_{\Lambda(a),\Lambda(b)} \otimes \iota)(x \otimes c)$, this implies that c is an element of $\overline{BC}$.
In a similar way, one proves that CB is dense in C. ∎

We would like to prove that $\mathcal{U}$ is an algebraic multiplier of $A \odot B$. For this, we need the following lemma.

Lemma 6.4 *Consider elements $a_1, \ldots, a_m, b_1, \ldots, b_m, c_1, \ldots, c_m$ and $p_1, \ldots, p_n, q_1, \ldots, q_n, r_1, \ldots, r_n$ in A such that $\sum_{i=1}^m a_i \otimes b_i \rho(c_i^*) = \sum_{j=1}^n \Delta(r_j \rho(s_j^*))(t_j \otimes 1)$. Then*

$$\sum_{i=1}^m \mathcal{U}\,(\pi_r(a_i) \otimes (\omega_{\Lambda(b_i),\Lambda(c_i)} \otimes \iota)(\mathcal{U})) = \sum_{j=1}^n \pi_r(t_j) \otimes (\omega_{\Lambda(r_j),\Lambda(s_j)} \otimes \iota)(\mathcal{U}) \ .$$

Proof : Using the fact that $(\Delta_r \otimes \iota)(\mathcal{U}) = \mathcal{U}_{13}\,\mathcal{U}_{23}$, we get that

$$\begin{aligned}
\sum_{i=1}^m \mathcal{U}\,(\pi_r(a_i) \otimes (\omega_{\Lambda(b_i),\Lambda(c_i)} \otimes \iota)(\mathcal{U})) &= (\iota \otimes \omega_{\Lambda(b_i),\Lambda(c_i)} \otimes \iota)(\mathcal{U}_{13}\,\mathcal{U}_{23}(\pi_r(a_i) \otimes 1 \otimes 1)) \\
&= (\iota \otimes \omega_{\Lambda(b_i),\Lambda(c_i)} \otimes \iota)((\Delta_r \otimes \iota)(\mathcal{U})(\pi_r(a_i) \otimes 1 \otimes 1)) \ . \qquad (*)
\end{aligned}$$

Choose $p \in A$ and $q \in C$. Then

$$\begin{aligned}
&\sum_{i=1}^m (\iota \otimes \omega_{\Lambda(b_i),\Lambda(c_i)} \otimes \iota)((\Delta_r \otimes \iota)(\pi_r(p) \otimes q)(\pi_r(a_i) \otimes 1 \otimes 1)) \\
&\quad = \sum_{i=1}^m (\iota \otimes \omega_{\Lambda(b_i),\Lambda(c_i)} \otimes \iota)((\pi_r \odot \pi_r)(\Delta(p)(a_i \otimes 1)) \otimes q) \\
&\quad = \sum_{i=1}^m (\pi_r \odot \varphi)((1 \otimes c_i^*)\Delta(p)(a_i \otimes b_i)) \otimes q \\
&\quad = \sum_{i=1}^m (\pi_r \odot \varphi)\big(\Delta(p)(a_i \otimes b_i \rho(c_i^*))\big) \otimes q \\
&\quad = \sum_{j=1}^n (\pi_r \odot \varphi)(\Delta(p\, r_j \rho(s_j^*))(t_j \otimes 1)) \otimes q \ .
\end{aligned}$$

Therefore, the left invariance of φ implies that

$$\begin{aligned}
&\sum_{i=1}^m (\iota \otimes \omega_{\Lambda(b_i),\Lambda(c_i)} \otimes \iota)((\Delta_r \otimes \iota)(\pi_r(p) \otimes q)(\pi_r(a_i) \otimes 1 \otimes 1)) \\
&\quad = \sum_{j=1}^n \pi_r(t_j)\, \varphi(p\, r_j \rho(s_j^*)) \otimes q = \sum_{j=1}^n \pi_r(t_j) \otimes \varphi(s_j^* p\, r_j)\, q \\
&\quad = \sum_{j=1}^n \pi_r(t_j) \otimes \omega_{\Lambda(r_j),\Lambda(s_j)}(\pi_r(p))\, q = \sum_{j=1}^n \pi_r(t_j) \otimes (\omega_{\Lambda(r_j),\Lambda(s_j)} \otimes \iota)(\pi_r(p) \otimes q) \ .
\end{aligned}$$

Hence, using equation (*), the usual continuity arguments imply that

$$\sum_{i=1}^{m} \mathcal{U}\,(\pi_r(a_i) \otimes (\omega_{\Lambda(b_i),\Lambda(c_i)} \otimes \iota)(\mathcal{U})) = \sum_{j=1}^{n} \pi_r(t_j) \otimes (\omega_{\Lambda(r_j),\Lambda(s_j)} \otimes \iota)(\mathcal{U}) \ .$$

■

This lemma implies immediately the following proposition (the second equality requires of course a lemma similar to the one above).

Proposition 6.5 *We have that* $\mathcal{U}\,(\pi_r(A) \odot B) = \pi_r(A) \odot B$ *and* $(\pi_r(A) \odot B)\,\mathcal{U} = \pi_r(A) \odot B$.

This justifies the following notation.

Notation 6.6 *We define* $\tilde{\mathcal{U}}$ *as the unique element in* $M(A \odot B)$ *such that* $(\pi_r \odot \iota)(x)\,\mathcal{U} = (\pi_r \odot \iota)(x\,\tilde{\mathcal{U}})$ *and* $\mathcal{U}\,(\pi_r \odot \iota)(x) = (\pi_r \odot \iota)(\tilde{\mathcal{U}}\,x)$ *for every* $x \in A \odot B$.

Proposition 6.7 *The element* $\tilde{\mathcal{U}}$ *is a non-degenerate corepresentation of* (A, Δ) *on* B.

Proof : We have for every $a \in A$, $x \in A \odot A$ and $b \in B$ that

$$\begin{aligned}
&(\pi_r \odot \pi_r \odot \iota)((\Delta \odot \iota)(\tilde{\mathcal{U}})(\Delta(a)\,x \otimes b)) = (\pi_r \odot \pi_r \odot \iota)((\Delta \odot \iota)(\tilde{\mathcal{U}}(a \otimes b))(x \otimes 1)) \\
&\quad = (\Delta_r \otimes \iota)\big((\pi_r \odot \iota)(\tilde{\mathcal{U}}(a \otimes b))\big)\,((\pi_r \odot \pi_r)(x) \otimes 1) = (\Delta_r \otimes \iota)(\mathcal{U}\,(\pi_r(a) \otimes b))\,((\pi_r \odot \pi_r)(x) \otimes 1) \\
&\quad = (\Delta_r \otimes \iota)(\mathcal{U})\,(\Delta_r(\pi_r(a))(\pi_r \odot \pi_r)(x) \otimes b) = \mathcal{U}_{13}\,\mathcal{U}_{23}\,((\pi_r \odot \pi_r)(\Delta(a)\,x) \otimes b) \\
&\quad = \mathcal{U}_{13}\,(\pi_r \odot \pi_r \odot \iota)(\tilde{\mathcal{U}}_{23}(\Delta(a)\,x \otimes b)) = (\pi_r \odot \pi_r \odot \iota)(\tilde{\mathcal{U}}_{13}\,\tilde{\mathcal{U}}_{23}(\Delta(a)\,x \otimes b))
\end{aligned}$$

which implies that $(\Delta \odot \iota)(\tilde{\mathcal{U}})(\Delta(a)\,x \otimes b) = \tilde{\mathcal{U}}_{13}\,\tilde{\mathcal{U}}_{23}(\Delta(a)\,x \otimes b)$. So we get that $(\Delta \odot \iota)(\tilde{\mathcal{U}}) = \tilde{\mathcal{U}}_{13}\,\tilde{\mathcal{U}}_{23}$. Proposition 6.5 of this section and proposition 6.10 of [11] imply immediately that $\tilde{\mathcal{U}}$ is non-degenerate.
■

Next, we want to show that B is a sub-*-algebra of $M(C)$. For this, we will use the next lemma.

Lemma 6.8 *We have for every* $a \in A$ *and* $b \in B$ *that*

$$\mathcal{U}^*(\pi_r(a) \otimes b) = (\pi_r \odot \iota)\big((S \odot \iota)((S^{-1}(a) \otimes 1)\,\tilde{\mathcal{U}}(1 \otimes b))\big) \ .$$

Proof : By proposition 5.11 of [11] we know that

$$\tilde{\mathcal{U}}\,(S \odot \iota)((S^{-1}(a) \otimes 1)\,\tilde{\mathcal{U}}(1 \otimes b)) = a \otimes b \ .$$

Therefore

$$\mathcal{U}\,(\pi_r \odot \iota)\big((S \odot \iota)((S^{-1}(a) \otimes 1)\,\tilde{\mathcal{U}}(1 \otimes b))\big) = (\pi_r \odot \iota)\big(\tilde{\mathcal{U}}\,(S \odot \iota)((S^{-1}(a) \otimes 1)\,\tilde{\mathcal{U}}(1 \otimes b))\big) = \pi_r(a) \otimes b$$

which implies that

$$(\pi_r \odot \iota)\big((S \odot \iota)((S^{-1}(a) \otimes 1)\,\tilde{\mathcal{U}}(1 \otimes b))\big) = \mathcal{U}^*(\pi_r(a) \otimes b) \ .$$

■

Proposition 6.9 *The set* B *is a sub-***-algebra of* $M(C)$ *such that* $\overline{BC} = C$.

Proof : Because of the previous results, we only have to prove that B is selfadjoint.
Choose $a,b,c \in A$. Take $y \in B$. Then

$$\begin{aligned}
&(\omega_{\Lambda(a),\Lambda(cb)} \otimes \iota)(\mathcal{U})^* \, y = (\omega_{\Lambda(cb),\Lambda(a)} \otimes \iota)(\mathcal{U}^*) \, y \\
&\quad = (\omega_{\pi_r(c)\Lambda(b),\Lambda(a)} \otimes \iota)(\mathcal{U}^*) \, y = (\omega_{\Lambda(b),\Lambda(a)} \otimes \iota)(\mathcal{U}^*(\pi_r(c) \otimes y)) \\
&\quad = (\omega_{\Lambda(b),\Lambda(a)} \otimes \iota)\big((\pi_r \odot \iota)((S \odot \iota)((S^{-1}(c) \otimes 1)\tilde{\mathcal{U}}(1 \otimes y))) \big) \qquad (*)
\end{aligned}$$

where we used the previous lemma in the last equality.

We have for every $p \in A$ and $q \in B$ that

$$\begin{aligned}
&(\omega_{\Lambda(b),\Lambda(a)} \otimes \iota)\big((\pi_r \odot \iota)((S \odot \iota)(p \otimes q))\big) = \omega_{\Lambda(b),\Lambda(a)}(\pi_r(S(p))) \; q \\
&\quad = \varphi(a^* S(p) b) \, q = \varphi(S(S^{-1}(b) \, p \, S^{-1}(a^*))) \, q = \varphi(S^{-1}(b) \, p \, S^{-1}(a^*) \, \delta) \, q \\
&\quad = \omega_{\Lambda(S^{-1}(a^*)\delta),\Lambda(S^{-1}(b)^*)}(\pi_r(p)) \, q = (\omega_{\Lambda(S^{-1}(a^*)\delta),\Lambda(S^{-1}(b)^*)} \otimes \iota)(\pi_r(p) \otimes q) \; .
\end{aligned}$$

Combining this with equality (*), we find that

$$\begin{aligned}
&(\omega_{\Lambda(a),\Lambda(cb)} \otimes \iota)(\mathcal{U})^* \, y = (\omega_{\Lambda(S^{-1}(a^*)\delta),\Lambda(S^{-1}(b)^*)} \otimes \iota)\big((\pi_r \odot \iota)((S^{-1}(c) \otimes 1)\,\tilde{\mathcal{U}}(1 \otimes y))\big) \\
&\quad = (\omega_{\Lambda(S^{-1}(a^*)\delta),\Lambda(S^{-1}(b)^*)} \otimes \iota)((\pi_r(S^{-1}(c)) \otimes 1)\,\mathcal{U}(1 \otimes y)) \\
&\quad = (\omega_{\Lambda(S^{-1}(a^*)\delta),\pi_r(S^{-1}(c)^*)\Lambda(S^{-1}(b)^*)} \otimes \iota)(\mathcal{U}(1 \otimes y)) \\
&\quad = (\omega_{\Lambda(S^{-1}(a^*)\delta),\Lambda(S^{-1}(cb)^*)} \otimes \iota)(\mathcal{U}) \, y \; .
\end{aligned}$$

So we see that $(\omega_{\Lambda(a),\Lambda(cb)} \otimes \iota)(\mathcal{U})^* = (\omega_{\Lambda(S^{-1}(a^*)\delta),\Lambda(S^{-1}(cb)^*)} \otimes \iota)(\mathcal{U})$ which clearly belongs to B.
Using the fat that $A^2 = A$, the proposition follows. ■

So we have proven in effect that $\mathcal{U}$ is of an algebraic nature. However, there is a *-algebra which is more natural than the *-algebra B. We will introduce this *-algebra in the rest of this section.

By the previous results, we have the following proposition.

Proposition 6.10 *The set BCB is a dense sub-*-algebra of C such that $(\pi_r(A) \odot BCB)\,\mathcal{U}$ $= \pi_r(A) \odot BCB$ and $\mathcal{U}\,(\pi_r(A) \odot BCB) = \pi_r(A) \odot BCB$.*

The following object seems to be a natural candidate for our special *-algebra connected with $\mathcal{U}$.

Definition 6.11 *We define the set*

$$C_{\mathcal{U}} = \{\, y \in C \mid \mathcal{U}(\pi_r(A) \otimes y) \subseteq \pi_r(A) \odot C \text{ and } (\pi_r(A) \otimes y)\,\mathcal{U} \subseteq \pi_r(A) \odot C \,\} \; .$$

It is easy to see that $C_{\mathcal{U}}$ is a subalgebra of C. The previous proposition implies that $BCB \subseteq C_{\mathcal{U}}$ which implies that $C_{\mathcal{U}}$ is dense in C. It is also easy to see that $BC_{\mathcal{U}}$ and $C_{\mathcal{U}}B$ are subsets of $C_{\mathcal{U}}$.

Proposition 6.12 *We have that $\mathcal{U}\,(\pi_r(A) \odot C_{\mathcal{U}}) \subseteq \pi_r(A) \odot C_{\mathcal{U}}$ and $(\pi_r(A) \odot C_{\mathcal{U}})\,\mathcal{U} \subseteq \pi_r(A) \odot C_{\mathcal{U}}$.*

Proof : Choose $a \in A$ and $x \in C_{\mathcal{U}}$.
By definition, there exists $b_1,\ldots,b_n \in A$ and $y_1,\ldots,y_n \in C$ such that $\mathcal{U}(\pi_r(a) \otimes x) = \sum_{i=1}^n \pi_r(b_i) \otimes y_i$.
Using the Gramm-Schmidt orthonormalization procedure, we can find elements $e_1,\ldots,e_m \in A$ such that $\varphi(e_j^* e_i) = \delta_{ij}$ for every $i,j \in \{1,\ldots,m\}$ and such that $b_1,\ldots,b_n$ belong to $\langle e_1,\ldots,e_m \rangle$.
Then there exist $z_1,\ldots,z_m \in C$ such that $\mathcal{U}(\pi_r(a) \otimes x) = \sum_{i=1}^m \pi_r(e_i) \otimes z_i$.

Fix $j \in \{1, \ldots, m\}$. There exist $d \in A$ such that $e_i d = e_i$ for every $i \in \{1, \ldots, m\}$.
We have that

$$\begin{aligned}
(\omega_{\Lambda(ad),\Lambda(e_j)} \otimes \iota)(\mathcal{U})\, x &= (\omega_{\pi_r(a)\Lambda(d),\Lambda(e_j)} \otimes \iota)(\mathcal{U})\, x = (\omega_{\Lambda(d),\Lambda(e_j)} \otimes \iota)(\mathcal{U}(\pi_r(a) \otimes x)) \\
&= \sum_{i=1}^{m} (\omega_{\Lambda(d),\Lambda(e_j)} \otimes \iota)(\pi_r(e_i) \otimes z_i) = \sum_{i=1}^{m} \varphi(e_j^* e_i d)\, z_i = \sum_{i=1}^{m} \varphi(e_j^* e_i)\, z_i = z_j \; .
\end{aligned}$$

Hence we see that z_j belongs to $BC_{\mathcal{U}}$ which implies that z_j belongs to $C_{\mathcal{U}}$.
So we get that $\mathcal{U}(\pi_r(a) \otimes x)$ belongs to $\pi_r(A) \odot C_{\mathcal{U}}$. Analogously, we get that $(\pi_r(a) \otimes x)\mathcal{U}$ belongs to $\pi_r(A) \odot C_{\mathcal{U}}$. ■

Proposition 6.13 *We have that $C_{\mathcal{U}} = BC_{\mathcal{U}} = C_{\mathcal{U}} B = BC_{\mathcal{U}} B$.*

Proof : Choose $x \in C_{\mathcal{U}}$. Take $a, b, c \in A$ such that $\varphi(c^* ab) = 1$. By the previous result, we know that $\mathcal{U}(\pi_r(a) \otimes x)$ belongs to $\pi_r(A) \odot C_{\mathcal{U}}$. So there exist $b_1, \ldots, b_n \in A$ and $y_1, \ldots, y_n \in C_{\mathcal{U}}$ such that $\mathcal{U}(\pi_r(a) \otimes x) = \sum_{i=1}^n \pi_r(b_i) \otimes y_i$.
Therefore, we get that $\pi_r(a) \otimes x = \sum_{i=1}^n \mathcal{U}^*(\pi_r(b_i) \otimes y_i)$ which implies that

$$\begin{aligned}
x &= \omega_{\Lambda(b),\Lambda(c)}(\pi_r(a))\, x = \sum_{i=1}^{n} (\omega_{\Lambda(b),\Lambda(c)} \otimes \iota)(\mathcal{U}^*(\pi_r(b_i) \otimes y_i)) \\
&= \sum_{i=1}^{n} (\omega_{\pi_r(b_i)\Lambda(b),\Lambda(c)} \otimes \iota)(\mathcal{U}^*(1 \otimes y_i)) = \sum_{i=1}^{n} (\omega_{\Lambda(b_i b),\Lambda(c)} \otimes \iota)(\mathcal{U}^*)\, y_i \\
&= \sum_{i=1}^{n} (\omega_{\Lambda(c),\Lambda(b_i b)} \otimes \iota)(\mathcal{U})^*\, y_i \; .
\end{aligned}$$

Because B is selfadjoint, this equation implies that x belongs to $BC_{\mathcal{U}}$.
So we have proven that $C_{\mathcal{U}} \subseteq BC_{\mathcal{U}}$ which implies that $C_{\mathcal{U}} = BC_{\mathcal{U}}$. Similarly, we get that $C_{\mathcal{U}} B = C_{\mathcal{U}}$. From this, we get immediately that $BC_{\mathcal{U}} B = C_{\mathcal{U}}$. ■

Corollary 6.14 *We have that $C_{\mathcal{U}} = BCB$.*

This follows immediately because $C_{\mathcal{U}} = BC_{\mathcal{U}} B \subseteq BCB \subseteq C_{\mathcal{U}}$.

Consequently, we arrive at the following conclusion.

Proposition 6.15 *The set $C_{\mathcal{U}}$ is a dense sub-*-algebra of C such that $\mathcal{U}(\pi_r(A) \odot C_{\mathcal{U}}) = \pi_r(A) \odot C_{\mathcal{U}}$ and $(\pi_r(A) \odot C_{\mathcal{U}})\mathcal{U} = \pi_r(A) \odot C_{\mathcal{U}}$.*

Remark 6.16 From the definition of $C_{\mathcal{U}}$, it is also immediately clear that $C_{\mathcal{U}}$ is the largest subspace of C such that $\mathcal{U}(\pi_r(A) \odot C_{\mathcal{U}}) \subseteq \pi_r(A) \odot C_{\mathcal{U}}$ and $(\pi_r(A) \odot C_{\mathcal{U}})\mathcal{U} \subseteq \pi_r(A) \odot C_{\mathcal{U}}$.

Now we can give the following definition :

Definition 6.17 *We define the element $\hat{\mathcal{U}} \in M(A \odot C_{\mathcal{U}})$ such that $(\pi_r \odot \iota)(x)\mathcal{U} = (\pi_r \odot \iota)(x\hat{\mathcal{U}})$ and $\mathcal{U}\,(\pi_r \odot \iota)(x) = (\pi_r \odot \iota)(\hat{\mathcal{U}}\, x)$ for every $x \in A \odot C_{\mathcal{U}}$. Then $\hat{\mathcal{U}}$ is a unitary corepresentation of (A, Δ) on $C_{\mathcal{U}}$.*

7 Lifting unitary corepresentations from the reduced to the universal C*-algebra

In section 5, we lifted certain *-automorphisms on A_r to *-automorphisms on A_u. Using the results of the previous section, we can now easily do a similar thing for unitary corepresentations of (A_r, Δ_r).

For the first part of this section, we fix a C*-algebra C and a unitary corepresentation $\mathcal{U}$ from (A_r, Δ_r) on C. This corepresentation will give rise to a corepresentation on (A_u, Δ_u).

We will use the notations $C_{\mathcal{U}}$ and $\hat{\mathcal{U}}$ from the previous section. So $\hat{\mathcal{U}}$ is an algebraic unitary corepresentation of (A, Δ) on $C_{\mathcal{U}}$. Then we can give immediately the following definition.

Definition 7.1 *We define $\mathcal{U}_u$ as the unitary element in $M(A_u \otimes C)$ such that $\mathcal{U}_u \, (\pi_u \odot \iota)(x) = (\pi_u \odot \iota)(\hat{\mathcal{U}}\, x)$ and $(\pi_u \odot \iota)(x)\, \mathcal{U}_u = (\pi_u \odot \iota)(x \hat{\mathcal{U}})$ for every $x \in A \odot C_{\mathcal{U}}$.*

Then the following proposition is an easy consequence.

Proposition 7.2 *The element $\mathcal{U}_u$ is a unitary corepresentation of (A_u, Δ_u) on C such that $(\pi \otimes \iota)(\mathcal{U}_u) = \mathcal{U}$.*

The following lemma guarantees that $\mathcal{U}_u$ is uniquely determined by the above property.

Lemma 7.3 *Consider a corepresentation $\mathcal{Q}$ of (A_u, Δ_u) on C such that $(\pi \otimes \iota)(\mathcal{Q}) = \mathcal{U}$. Then we have that $\mathcal{Q}_{13} = V_{12}^* \mathcal{U}_{23} V_{12} \mathcal{U}_{23}^*$.*

Proof : Because $\mathcal{Q}$ is a corepresentation of (A_u, Δ_u) on C, we have that $(\Delta_u \otimes \iota)(\mathcal{Q}) = \mathcal{Q}_{13}\mathcal{Q}_{23}$. If we apply $\iota \otimes \pi \otimes \iota$ to this equation and use the fact that $(\pi \otimes \iota)(\mathcal{Q}) = \mathcal{U}$, we get that

$$((\iota \otimes \pi)\Delta_u \otimes \iota)(\mathcal{Q}) = \mathcal{Q}_{13}\, \mathcal{U}_{23} \ .$$

By proposition 4.13, we know that $(\iota \otimes \pi)\Delta_u(x) = V^*(1 \otimes \pi(x))V$ for every $x \in A_u$. This implies that

$$((\iota \otimes \pi)\Delta_u \otimes \iota)(\mathcal{Q}) = V_{12}^* (\pi \otimes \iota)(\mathcal{Q})_{23} V_{12} = V_{12}^* \mathcal{U}_{23} V_{12} \ .$$

So we get that $V_{12}^* \mathcal{U}_{23} V_{12} = \mathcal{Q}_{13}\, \mathcal{U}_{23}$. ■

Using this lemma, we get the following uniqueness result.

Proposition 7.4 *The element $\mathcal{U}_u$ is the unique corepresentation of (A_u, Δ_u) on C such that $(\pi \otimes \iota)(\mathcal{U}_u) = \mathcal{U}$.*

Reformulating lemma 7.3 with $\mathcal{Q}$ equal to $\mathcal{U}_u$, the following equality holds.

Result 7.5 *We have that $(\mathcal{U}_u)_{13} = V_{12}^* \mathcal{U}_{23} V_{12} \mathcal{U}_{23}^*$.*

Corollary 7.6 *We have for every $\omega \in B_0(H)$ that $((\iota \otimes \omega)(V) \otimes 1)\, \mathcal{U}_u = (\iota \otimes \omega \otimes \iota)(\mathcal{U}_{23} V_{12} \mathcal{U}_{23}^*)$.*

By definition 7.1, we have immediately the following result.

Result 7.7 *We have that $\mathcal{U}_u \, (\pi_u(A) \odot C_{\mathcal{U}}) = \pi_u(A) \odot C_{\mathcal{U}}$ and $(\pi_u(A) \odot C_{\mathcal{U}})\, \mathcal{U}_u = \pi_u(A) \odot C_{\mathcal{U}}$.*

Combining this with the fact that $(\pi \otimes \iota)(\mathcal{U}_u) = \mathcal{U}$ and definition 6.11, this implies the following result.

Result 7.8 *We have that*

$$C_{\mathcal{U}} = \{\, y \in C \mid \mathcal{U}_u(\pi_u(A) \otimes y) \subseteq \pi_u(A) \odot C \text{ and } (\pi_u(A) \otimes y)\mathcal{U}_u \subseteq \pi_u(A) \odot C \,\} \ .$$

Now we state the fact that there is a bijective correspondence between unitary corepresentations of (A_r, Δ_r) and (A_u, Δ_u).

Theorem 7.9 *Consider a C^*-algebra C, then we have the following two properties.*

1. *Let $\mathcal{U}$ be a unitary corepresentation of (A_r, Δ_r) on C. Then $\mathcal{U}_u$ is the unique unitary corepresentation of (A_u, Δ_u) on C such that $(\pi \otimes \iota)(\mathcal{U}_u) = \mathcal{U}$.*
2. *Let $\mathcal{U}$ be a unitary corepresentation of (A_u, Δ_u) on C. Then $(\pi \otimes \iota)(\mathcal{U})$ is the unique unitary corepresentation of (A_r, Δ_r) on C such that $[(\pi \otimes \iota)(\mathcal{U})]_u = \mathcal{U}$.*

Proof :

1. This is just repeating proposition 7.4.
2. By proposition 7.4, we know that $[(\pi \otimes \iota)(\mathcal{U})]_u$ is the unique unitary corepresentation of (A_u, Δ_u) on C such that $(\pi \otimes \iota)\big([(\pi \otimes \iota)(\mathcal{U})]_u\big) = (\pi \otimes \iota)(\mathcal{U})$. So we must have that $[(\pi \otimes \iota)(\mathcal{U})]_u = \mathcal{U}$.

   If $\mathcal{V}$ is a unitary corepresentation of (A_r, Δ_r) on C such that $\mathcal{V}_u = (\pi \otimes \iota)(\mathcal{U})$, then proposition 7.2 implies that $\mathcal{V} = (\pi \otimes \iota)(\mathcal{V}_u) = (\pi \otimes \iota)(\mathcal{U})$.

■

If we combine the second statement of this proposition with results 7.7 and 7.8, we get the followin proposition.

Proposition 7.10 *Consider a unitary corepresentation $\mathcal{U}$ of (A_u, Δ_u) on a C^*-algebra C and define the set*

$$C_{\mathcal{U}} = \{\, y \in C \mid \mathcal{U}(\pi_u(A) \otimes y) \subseteq \pi_u(A) \odot C \text{ and } (\pi_u(A) \otimes y)\mathcal{U} \subseteq \pi_u(A) \odot C \,\} \ .$$

Then $C_{\mathcal{U}}$ is a dense sub-*-algebra of C such that $\mathcal{U}\,(\pi_u(A) \odot C_{\mathcal{U}}) = \pi_u(A) \odot C_{\mathcal{U}}$ and $(\pi_u(A) \odot C_{\mathcal{U}})\mathcal{U} = \pi_u(A) \odot C_{\mathcal{U}}$.

Of course, this proposition can also be proven in the same way as we did for unitary corepresentations of (A_r, Δ_r) in the previous section.

So we can give the following definition.

Definition 7.11 *Consider a unitary corepresentation $\mathcal{U}$ of (A_u, Δ_u) on a C^*-algebra C. Then we define the element $\hat{\mathcal{U}} \in M(A \odot C_{\mathcal{U}})$ such that $(\pi_u \odot \iota)(x)\mathcal{U} = (\pi_u \odot \iota)(x\hat{\mathcal{U}})$ and $\mathcal{U}\,(\pi_u \odot \iota)(x) = (\pi_u \odot \iota)(\hat{\mathcal{U}}\,x)$ for every $x \in A \odot C_{\mathcal{U}}$. Then $\hat{\mathcal{U}}$ is a unitary corepresentation of (A, Δ) on $C_{\mathcal{U}}$.*

The first statement of the next proposition follows from result 7.8, definition 7.1 and the previous definition. Using theorem 7.9, the second one follows from the first.

Proposition 7.12 *Consider a C^*-algebra C, then we have the following properties.*

1. *Consider a unitary corepresentation $\mathcal{U}$ of (A_r, Δ_r) on C. Then we have that $C_{\mathcal{U}_u} = C_{\mathcal{U}}$ and $(\mathcal{U}_u)\hat{} = \hat{\mathcal{U}}$*
2. *Consider a unitary corepresentation $\mathcal{U}$ of (A_u, Δ_u) on C. Then we have that $C_{(\pi\otimes\iota)(\mathcal{U})} = C_{\mathcal{U}}$ and $\big((\pi \otimes \iota)(\mathcal{U})\big)\hat{} = \hat{\mathcal{U}}$*

8 The modular group of the left Haar weight on (A_u, Δ_u)

In a later section, we will introduce the left Haar weight on (A_u, Δ_u) using the left Haar weight on (A_r, Δ_r). We want this left Haar weight on A_u to be a KMS-weight with respect to some one-parameter group. In this section, we will introduce this one-parameter group using the techniques of section 5.

Remember from section 2, that we have the left Haar weight φ_r on (A_r, Δ_r). This weight is a KMS-weight and the modular group of φ_r was denoted by σ_r. For some more notations, we refer to the remarks after proposition 2.15.
Then we have that φ_r is invariant under σ_r and that $\Lambda_r((\sigma_r)_t(a)) = \nabla^{it}\Lambda_r(a)$ for every $a \in \mathcal{N}_{\varphi_r}$.
We know that $\Lambda(a)$ is analytic with respect to ∇ and that $\nabla^n\Lambda(a) = \Lambda(\rho^n(a))$ for every $a \in A$ and $n \in \mathbb{Z}$.

By definition 3.14 of [9], there exist a unique norm continuous one-parameter group K_r on A_r such that

$$((\sigma_r)_t \otimes (K_r)_t)\Delta_r = \Delta_r\,(\sigma_r)_t$$

for every $t \in \mathbb{R}$. Notice that in [9], K_r was denoted by K. Proposition 5.3 implies that φ_r is invariant with respect to K_r.

In section 3 of [9], we introduced a positive injective operator P in H which implements K_r, i.e. $(K_r)_t(x) = P^{it}xP^{-it}$ for every $x \in A_r$ and $t \in \mathbb{R}$.

Result 8.1 *We have for every $t \in \mathbb{R}$ and $a \in \mathcal{N}_{\varphi_r}$ that $P^{it}\Lambda_r(a) = \Lambda_r((K_r)_t(a))$.*

Proof : Fix $t \in \mathbb{R}$ and define the unitary operator u on H such that $u\Lambda_r(a) = \Lambda_r((K_r)_t(a))$ for every $a \in \mathcal{N}_{\varphi_r}$.
Then proposition 5.7 implies that $(\nabla^{it} \otimes \nabla^{it})W = W(\nabla^{it} \otimes u)$. At the same time, proposition 3.12 of [9] implies that $(\nabla^{it} \otimes \nabla^{it})W = W(\nabla^{it} \otimes P^{it})$. Comparing these two results, we see that $W(1 \otimes P^{it}) = W(1 \otimes u)$. Consequently, $u = P^{it}$. ■

Result 8.2 *Consider $a \in A$ and $n \in \mathbb{Z}$. Then $\Lambda(a)$ belongs to $D(P^n)$ and $P^n\Lambda(a) = \Lambda(\delta^{-n}S^{-2n}(a)\delta^n)$.*

The case $n = 0$ is trivially true. The case $n = 1$ follows from the remark after definition 3.6 of [9]. The case $n = -1$ follows from the case $n = 1$. The result follows now by induction.

Corollary 8.3 *Consider $a \in A$. Then $\Lambda(a)$ is analytic with respect to P.*

By the results of section 5, we have for every $t \in \mathbb{R}$ the *-automorphisms $(\sigma_t)_u$ and $(K_t)_u$ on A_u such that

- $(\sigma_t)_u((\iota \otimes \omega)(V)) = (\iota \otimes P^{it}\omega\nabla^{-it})(V)$
- $(K_t)_u((\iota \otimes \omega)(V)) = (\iota \otimes P^{it}\omega P^{-it})(V)$

for every $\omega \in B_0(H)^*$.

From the previous formulas, we get the following results :

1. We have for every $a \in A_u$ that the mapping $\mathbb{R} \to A_u : t \mapsto (\sigma_t)_u(a)$ is norm continuous.
2. We have for every $s, t \in \mathbb{R}$ that $(\sigma_s)_u\,(\sigma_t)_u = (\sigma_{s+t})_u$.

So we have the following definition.

Definition 8.4 *We define the norm continuous one-parameter group σ_u on A_u such that $(\sigma_u)_t = (\sigma_t)_u$ for every $t \in \mathbb{R}$.*

Hence, σ_u is determined by the property that

$$(\sigma_u)_t((\iota \otimes \omega)(V)) = (\iota \otimes P^{it}\omega\nabla^{-it})(V)$$

for every $\omega \in B_0(H)^*$ and $t \in \mathbb{R}$.

Result 5.22 implies that $\pi\,(\sigma_u)_t = (\sigma_r)_t\,\pi$ for every $t \in \mathbb{R}$.

In a similar way, we can give the following definition.

Definition 8.5 *We define the norm continuous one-parameter group K_u on A_u such that $(K_u)_t = (K_t)_u$ for every $t \in \mathbb{R}$.*

Again, K_u is determined by the property that

$$(K_u)_t((\iota \otimes \omega)(V)) = (\iota \otimes P^{it}\omega P^{-it})(V)$$

for every $\omega \in B_0(H)^*$.

We have also that $\pi\,(K_u)_t = (K_r)_t\,\pi$ for every $t \in \mathbb{R}$.

Proposition 5.21 implies the following commutation relation.

Result 8.6 *We have for every $t \in \mathbb{R}$ that $((\sigma_u)_t \otimes (K_u)_t)\Delta_u = \Delta_u\,(\sigma_u)_t$.*

Later on, we shall show that σ_u and K_u are determined by the properties above (see proposition 11.8).

In the rest of this section, we want to show that every element of $\pi_u(A)$ is analytic with respect to σ_u.

Lemma 8.7 *Let b, c be elements in A. Then $\rho^{-1}\big((\iota\odot\varphi)(\Delta(c^*)(1\otimes b))\big) = (\iota\odot\varphi)(\Delta(\rho(c)^*)(1\otimes\delta S^2(b)\delta^{-1}))$.*

Proof : By section 1, we have for every $x \in A$ that $\rho^{-1}(x) = \delta\,(\rho')^{-1}(x)\,\delta^{-1}$. This implies that

$$\begin{aligned}(\rho^{-1} \odot \iota)(\Delta(c^*)(1 \otimes b)) &= (\delta \otimes 1)\,((\rho')^{-1} \odot \iota)(\Delta(c^*))\,(\delta^{-1} \otimes b) \\ &= (\delta \otimes 1)\,(\iota \odot S^{-2})\big(\Delta((\rho')^{-1}(c^*))\big)\,(\delta^{-1} \otimes b)\end{aligned}$$

where we used the results of section 1 in the last equality. Using the fact that $(\rho')^{-1}(c^*) = \delta^{-1}\rho^{-1}(c^*)\delta$, this implies that

$$\begin{aligned}(\rho^{-1} \odot \iota)(\Delta(c^*)(1 \otimes b)) &= (\delta \otimes 1)\,(\iota \odot S^{-2})(\Delta(\delta^{-1}\rho^{-1}(c^*)\delta))\,(\delta^{-1} \otimes b) \\ &= (1 \otimes \delta^{-1})\,(\iota \odot S^{-2})(\Delta(\rho(c)^*))\,(1 \otimes \delta b) = (1 \otimes \delta^{-1})\,(\iota \odot S^{-2})(\Delta(\rho(c)^*)(1 \otimes \delta S^2(b)\delta^{-1}))\,(1 \otimes \delta)\end{aligned}$$

where we used that $\Delta(\delta) = \delta \otimes \delta$ and $S^{-2}(\delta) = \delta$. By section 1, we know that $\varphi(\delta^{-1}S^{-2}(x)\delta) = \varphi(x)$ for every $x \in A$. Therefore,

$$\begin{aligned}\rho^{-1}\big((\iota \odot \varphi)(\Delta(b^*)(1 \otimes c))\big) &= (\iota \odot \varphi)\big((\rho^{-1} \odot \iota)(\Delta(b^*)(1 \otimes c))\big) \\ &= (\iota \odot \varphi)((1 \otimes \delta^{-1})\,(\iota \odot S^{-2})(\Delta(\rho(c)^*)(1 \otimes \delta S^2(b)\delta^{-1}))\,(1 \otimes \delta)) \\ &= (\iota \odot \varphi)(\Delta(\rho(c)^*)(1 \otimes \delta S^2(b)\delta^{-1}))\ .\end{aligned}$$

■

Proposition 8.8 *Consider $a \in A$. Then $\pi_u(a)$ is analytic with respect to σ_u and $(\sigma_u)_{ni}(\pi_u(a)) = \pi_u(\rho^{-n}(a))$ for every $n \in \mathbb{Z}$.*

Proof : Choose $b, c \in A$ and put $d = (\iota \odot \varphi)(\Delta(c^*)(1 \otimes b))$. Then lemma 4.3 implies that

$$\pi_u(d) = (\iota \otimes \omega_{\Lambda(b),\Lambda(c)})(V) .$$

Therefore, the remarks after definition 8.4 implies for every $t \in \mathbb{R}$ that

$$(\sigma_u)_t(\pi_u(d)) = (\iota \otimes \omega_{P^{it}\Lambda(b),\nabla^{it}\Lambda(c)})(V) \qquad (*)$$

Because $\Lambda(b)$ is analytic with respect to P and $\Lambda(c)$ is analytic with respect to ∇, this implies easily that $\pi_u(d)$ is analytic with respect to σ_u.

By equation (*), we have immediately that

$$(\sigma_u)_i(\pi_u(d)) = (\iota \otimes \omega_{P^{ii}\Lambda(b),\nabla^{i\,\bar{i}}\Lambda(c)})(V) = (\iota \otimes \omega_{P^{-1}\Lambda(b),\nabla\Lambda(c)})(V) .$$

Therefore, the remarks of the beginning of the section and result 8.2 imply that

$$(\sigma_u)_i(\pi_u(d)) = (\iota \otimes \omega_{\Lambda(\delta S^2(b)\delta^{-1}),\Lambda(\rho(c))})(V) .$$

Hence, using lemma 4.3 once more, we get that

$$(\sigma_u)_i(\pi_u(d)) = \pi_u\big((\iota \odot \varphi)(\Delta(\rho(c)^*)(1 \otimes \delta S^2(b)\delta^{-1}))\big) .$$

Therefore, the previous lemma implies that

$$(\sigma_u)_i(\pi_u(d)) = \pi_u\big(\,\rho^{-1}(\,(\iota \odot \varphi)(\Delta(c^*)(1 \otimes b))\,)\,\big) = \pi_u(\rho^{-1}(d)) .$$

From this all, we conclude for every $x \in A$ that $\pi_u(x)$ is analytic with respect to σ_u and that $(\sigma_u)_i(\pi_u(x)) = \pi_u(\rho^{-1}(x))$. The general result can now be proven easily. ∎

9 The polar decomposition of the antipode on A_u

In section 5 of [9], we arrived at a polar decomposition of the antipode on (A_r, Δ_r). This polar decomposition consists of a norm continuous one-parameter group τ_r on A_r (denoted by τ in [9]) and an involutive anti-*-automorphism R_r on A_r (denoted by R in [9]).
In this section, we want to transform these objects into similar objects on A_u and arrive at a polar decomposition of the antipode on A_u.

By proposition 5.8 of [9], we have for every $t \in \mathbb{R}$ that $((\tau_r)_t \otimes (\tau_r)_t)\Delta_r = \Delta_r\,(\tau_r)_t$. In corollary 7.3 of [9], we proved moreover the existence of a unique strictly positive number ν such that $\varphi\,(\tau_r)_t = \nu^t\,\varphi$ for every $t \in \mathbb{R}$.

Define the positive injective operator Q in H such that $Q^{it}\Lambda_r(a) = \nu^{-\frac{t}{2}}\,\Lambda_r((\tau_r)_t(a))$ for every $a \in \mathcal{N}_{\varphi_r}$. So we have that $(\tau_r)_t(x) = Q^{it}xQ^{-it}$ for every $x \in A_r$ and $t \in \mathbb{R}$.
We can now use again the results of section 5. So we have for every $t \in \mathbb{R}$ a *-automorphism $(\tau_t)_u$ on A_u such that $(\tau_t)_u((\iota \otimes \omega)(V)) = (\iota \otimes Q^{it}\omega Q^{-it})(V)$ for every $t \in \mathbb{R}$.

In the same way as in the previous section, this justifies the following definition.

Definition 9.1 *We define the norm continuous one-parameter group τ_u on A_u such that $(\tau_u)_t = (\tau_t)_u$ for every $t \in \mathbb{R}$.*

Hence, τ_u is determined by the property that

$$(\tau_u)_t((\iota \otimes \omega)(V)) = (\iota \otimes Q^{it}\omega Q^{-it})(V)$$

for every $\omega \in B_0(H)^*$ and $t \in \mathbb{R}$.

Again, result 5.22 implies that $\pi\,(\tau_u)_t = (\tau_r)_t\,\pi$ for every $t \in \mathbb{R}$.

From proposition 5.21, we get the following commutation relation.

Result 9.2 *We have for every $t \in \mathbb{R}$ that $((\tau_u)_t \otimes (\tau_u)_t)\Delta_u = \Delta_u\,(\tau_u)_t$.*

We would also like to obtain a formula like in equation (8) of section 5 of [9] for τ_u.

Result 9.3 *We have for every $t \in \mathbb{R}$ and $\omega \in B_0(H)^*$ that $(\tau_u)_t((\iota \otimes \omega)(V)) = (\iota \otimes \nabla^{it}\omega\nabla^{-it})(V)$.*

Proof : Fix $t \in \mathbb{R}$ and $\omega \in B_0(H)^*$. By proposition 5.7, we have that $(Q^{it} \otimes Q^{it})W = W(Q^{it} \otimes Q^{it})$. Hence, using equation (8) of [9] and the fact that τ_r is implemented by Q, we get that

$$(\iota \otimes \nabla^{it}\omega\nabla^{-it})(W) = (\tau_r)_t((\iota \otimes \omega)(W)) = Q^{it}(\iota \otimes \omega)(W)Q^{-it} = (\iota \otimes Q^{it}\omega Q^{-it})(W)\ .$$

The previous equality implies for every $\theta \in B_0(H)^*$ that

$$(\nabla^{it}\omega\nabla^{-it})((\theta \otimes \iota)(W)) = (Q^{it}\omega Q^{-it})((\theta \otimes \iota)(W))\ .$$

This implies that $(\nabla^{it}\omega\nabla^{-it})(x) = (Q^{it}\omega Q^{-it})(x)$ for every $x \in \hat{A}_r$. As a consequence, we get that $(\nabla^{it}\omega\nabla^{-it})(x) = (Q^{it}\omega Q^{-it})(x)$ for every $x \in M(\hat{A}_r)$.

Choose $\eta \in A_u^*$. Because V is an element of $M(A_u \otimes \hat{A}_r)$, we have that $(\eta \otimes \iota)(V)$ belongs to $M(\hat{A}_r)$. So the first part implies that

$$(\nabla^{it}\omega\nabla^{-it})((\eta \otimes \iota)(V)) = (Q^{it}\omega Q^{-it})((\eta \otimes \iota)(V))\ .$$

Hence,

$$\eta((\iota \otimes \nabla^{it}\omega\nabla^{-it})(V)) = \eta((\iota \otimes Q^{it}\omega Q^{-it})(V))\ .$$

So we see that

$$(\iota \otimes \nabla^{it}\omega\nabla^{-it})(V) = (\iota \otimes Q^{it}\omega Q^{-it})(V) = (\tau_u)_t((\iota \otimes \omega)(V))\ .$$

∎

Using the equalities $((\sigma_u)_t \otimes \iota)(V) = (1 \otimes \nabla^{-it})V(1 \otimes P^{it})$ and $((\tau_u)_t \otimes \iota)(V) = (1 \otimes \nabla^{-it})V(1 \otimes \nabla^{it})$, the proof of the next result is the same as the proof of proposition 5.21.

Result 9.4 *We have for every $t \in \mathbb{R}$ that $((\tau_u)_t \otimes (\sigma_u)_t)\Delta_u = \Delta_u\,(\sigma_u)_t$.*

In the next part of this section, we prove that every element of $\pi_u(A)$ is analytic with respect to τ_u. The idea behind the proof is the same as that of the proof of proposition 8.8.

Lemma 9.5 *Consider $a \in A$ and $n \in \mathbb{Z}$. Then $\Lambda(a)$ belongs to $D(Q^n)$ and $Q^n\Lambda(a) = \nu^{\frac{ni}{2}}\Lambda(S^{2n}(a))$.*

Proof : Choose $b \in A$. Then $\pi_r(b)$ belongs to $\mathcal{N}_{\varphi_r}$. We also know that $\pi_r(b)$ belongs to $\mathcal{D}((\tau_r)_{-i})$ and $(\tau_r)_{-i}(\pi_r(b)) = \pi_r(S^2(b))$ (see proposition 5.5 of [9]). So we get that $(\tau_r)_{-i}(\pi_r(b))$ belongs to $\mathcal{N}_{\varphi_r}$. Because $\Lambda_r((\tau_r)_t(x)) = \nu^{\frac{t}{2}}\, Q^{it} \Lambda_r(x)$ for every $x \in \mathcal{N}_{\varphi_r}$, this implies that $\Lambda_r(\pi_r(b))$ belongs to $D(Q)$ and

$$\nu^{-\frac{i}{2}}\, Q \Lambda_r(\pi_r(b)) = \Lambda_r\big((\tau_r)_{-i}(\pi_r(b))\big) = \Lambda_r\big(\pi_r(S^2(b))\big)$$

Because $\Lambda_r(\pi_r(c)) = \Lambda(c)$ for every $c \in A$, we see that $\Lambda(b)$ belongs to $D(Q)$ and $Q\Lambda(b) = \nu^{\frac{i}{2}}\, \Lambda(S^2(b))$. The result follows from this result by induction. ∎

Corollary 9.6 *Consider $a \in A$. Then $\Lambda(a)$ is analytic with respect to Q.*

Proposition 9.7 *Consider $a \in A$. Then $\pi_u(a)$ is analytic with respect to τ_u and $(\tau_u)_{ni}(\pi_u(a)) = \pi_u(S^{-2n}(a))$ for every $n \in \mathbb{Z}$.*

Proof : Choose $b, c \in A$ and put $d = (\iota \odot \varphi)(\Delta(c^*)(1 \otimes b))$. Again, lemma 4.3 implies that $\pi_u(d) = (\iota \otimes \omega_{\Lambda(b),\Lambda(c)})(V)$. Therefore, we get for every $t \in \mathbb{R}$ that

$$(\tau_u)_t(\pi_u(d)) = (\iota \otimes \omega_{Q^{it}\Lambda(b),Q^{it}\Lambda(c)})(V) \; . \qquad (*)$$

Because $\Lambda(b), \Lambda(c)$ are analytic with respect to Q, this implies immediately that $\pi_u(d)$ is analytic with respect to τ_u.
Furthermore, equality (*) implies also that

$$\begin{aligned}(\tau_u)_i(\pi_u(d)) &= (\iota \otimes \omega_{Q^{ii}\Lambda(b),Q^{i\,\bar{i}}\Lambda(c)})(V) = (\iota \otimes \omega_{Q^{-1}\Lambda(b),Q\Lambda(c)})(V) \\ &= \nu^{-i}\, (\iota \otimes \omega_{\Lambda(S^{-2}(b)),\Lambda(S^2(c))})(V)\end{aligned}$$

where we used lemma 9.5 in the last equality. Hence, lemma 4.3 implies that

$$\begin{aligned}(\tau_u)_i(\pi_u(d)) &= \nu^{-i}\, \pi_u\big(\,(\iota \odot \varphi)(\,\Delta(S^2(c)^*)(1 \otimes S^{-2}(b))\,)\,\big) \\ &= \nu^{-i}\, \pi_u\big(\,(\iota \odot \varphi)(\,\Delta(S^{-2}(c^*))(1 \otimes S^{-2}(b))\,)\,\big) \\ &= \nu^{-i}\, \pi_u\big(\,(\iota \otimes \varphi)(\,(S^{-2} \odot S^{-2})(\Delta(c^*)(1 \otimes b))\,)\,\big) \; .\end{aligned}$$

By corollary 8.19 of [9], we know that $\varphi S^{-2} = \mu^{-1}\, \varphi = \nu^i\, \varphi$. This implies that

$$(\tau_u)_i(\pi_u(d)) = \pi_u\big(\,S^{-2}(\,(\iota \odot \varphi)(\Delta(c^*)(1 \otimes b))\,)\,\big) = \pi_u(S^{-2}(d)) \; .$$

From this all, we conclude for all $x \in A$ that $\pi_u(x)$ is analytic with respect to τ_u and that $(\tau_u)_i(\pi_u(x)) = \pi_u(S^{-2}(x))$. The general result follows by induction. ∎

In the following part of this section, we will also transform R_r to a *-automorphism on A_u. The idea behind this procedure will be the same as the idea in section 5, but the details are somewhat different (for instance, φ_r is not relatively invariant under R_r).

At the end of section 7 of [9], we introduced the anti-unitary involution $\hat{J}$ on H. This is in fact the modular conjugation of the right invariant weight on $(\hat{A}_r, \hat{\Delta}_r)$. Proposition 7.15 of [9] guarantees that $(\hat{J} \otimes J)W^*(\hat{J} \otimes J) = W$. This relation was then used to prove that $R_r(x) = \hat{J}x^*\hat{J}$ for every $x \in A_r$.

As before, the relation $(\hat{J} \otimes J)W^*(\hat{J} \otimes J) = W$ will imply that $J\hat{A}_rJ = \hat{A}_r$ and $JM(\hat{A}_r)J = M(\hat{A}_r)$. Then we can use J to implement the anti-unitary antipode $\hat{R}_r$ on $(\hat{A}_r, \hat{\Delta}_r)$, i.e. $\hat{R}_r(x) = Jx^*J$ for every $x \in \hat{A}_r$. We will have of course also that $\hat{R}_r$ is an involutive anti-*-automorphism of $\hat{A}_r$ and that $\chi(\hat{R}_r \otimes \hat{R}_r)\hat{\Delta}_r = \hat{\Delta}_r\, \hat{R}_r$.

The proof of the following lemma is similar to the proof of lemma 5.11 .

Lemma 9.8 *Consider $\omega_1, \omega_2 \in B_0(H)^*$ such that $(\iota \otimes \omega_1)(V) = (\iota \otimes \omega_2)(V)$. Then $(\iota \otimes \omega_1(J.^*J))(V) = (\iota \otimes \omega_2(J.^*J))(V)$.*

As before, this justifies the following definition :

Proposition 9.9 *There exists a unique linear mapping F from $\pi_u(A)$ into A_u such that $F((\iota \otimes \omega_{p,q})(V)) = (\iota \otimes \omega_{Jq,Jp})(V)$ for every $p, q \in \Lambda(A)$.*

Result 9.10 *The mapping F is antimultiplicative.*

Proof : Choose $p_1, q_1, p_2, q_2 \in \Lambda(A)$. Define the mapping $\omega \in B_0(H)^*$ such that $\omega(x) = \langle W(x \otimes 1)W^*(p_1 \otimes p_2), q_1 \otimes q_2 \rangle$ for every $x \in B_0(H)$. As usual, we have that

$$(\iota \otimes \omega_{p_1,q_1})(V)\,(\iota \otimes \omega_{p_2,q_2})(V) = (\iota \otimes \omega)(V) \ .$$

Using lemma 9.8, this gives us that

$$F\big(\,(\iota \otimes \omega_{p_1,q_1})(V)\,(\iota \otimes \omega_{p_2,q_2})(V)\,\big) = (\iota \otimes \omega(J.^*J))(V) \ .$$

We have for every $x \in \hat{A}_r$ that

$$\begin{aligned} \omega(Jx^*J) &= \langle W(Jx^*J \otimes 1)W^*(p_1 \otimes p_2), q_1 \otimes q_2 \rangle = \langle W(\hat{R}_r(x) \otimes 1)W^*(p_1 \otimes p_2), q_1 \otimes q_2 \rangle \\ &= \langle \hat{\Delta}_r(\hat{R}_r(x))(p_1 \otimes p_2), q_1 \otimes q_2 \rangle = \langle \chi\big((\hat{R}_r \otimes \hat{R}_r)(\hat{\Delta}_r(x))\big)\,(p_1 \otimes p_2), q_1 \otimes q_2 \rangle \\ &= \langle (\hat{R}_r \otimes \hat{R}_r)(\hat{\Delta}_r(x))(p_2 \otimes p_1), q_2 \otimes q_1 \rangle = \langle (J \otimes J)\hat{\Delta}_r(x)^*(J \otimes J)(p_2 \otimes p_1), q_2 \otimes q_1 \rangle \\ &= \langle \hat{\Delta}_r(x)(Jq_2 \otimes Jq_1), Jp_2 \otimes Jp_1 \rangle = \langle W(x \otimes 1)W^* Jq_2 \otimes Jq_1, Jp_2 \otimes Jp_1 \rangle \ . \end{aligned}$$

Because V belongs to $M(A_u \otimes \hat{A}_r)$ this equality implies that

$$(\iota \otimes \omega(J.^*J))(V) = (\iota \otimes \omega_{Jq_2,Jp_2})(V)\,(\iota \otimes \omega_{Jq_1,Jp_1})(V) \ .$$

Hence,

$$\begin{aligned} F\big(\,(\iota \otimes \omega_{p_1,q_1})(V)\,(\iota \otimes \omega_{p_2,q_2})(V)\,\big) &= (\iota \otimes \omega_{Jq_2,Jp_2})(V)\,(\iota \otimes \omega_{Jq_1,Jp_1})(V) \\ &= F((\iota \otimes \omega_{p_2,q_2})(V))\,F((\iota \otimes \omega_{p_1,q_1})(V)) \ . \end{aligned}$$

The result follows by linearity. ■

Result 9.11 *The mapping F is selfadjoint.*

Proof : Choose $p, q \in \Lambda(A)$. Then $(\iota \otimes \omega_{p,q})(V)^* = (\iota \otimes \omega_{q,p})(V^*)$. Because $p \in D(\nabla^{-\frac{1}{2}})$ and $q \in D(\nabla^{\frac{1}{2}})$, corollary 4.16 implies that

$$(\iota \otimes \omega_{p,q})(V)^* = (\iota \otimes \omega_{J\nabla^{-\frac{1}{2}}p,\,J\nabla^{\frac{1}{2}}q})(V) \ .$$

Hence, lemma 9.8 implies that

$$F((\iota \otimes \omega_{p,q})(V)^*) = (\iota \otimes \omega_{\nabla^{\frac{1}{2}}q,\,\nabla^{-\frac{1}{2}}p})(V) \ .$$

We also have that Jq belongs to $D(\nabla^{-\frac{1}{2}})$ and $\nabla^{-\frac{1}{2}}Jq = J\nabla^{\frac{1}{2}}q$ and that Jp belongs to $D(\nabla^{\frac{1}{2}})$ and $\nabla^{\frac{1}{2}}Jp = J\nabla^{-\frac{1}{2}}p$. So we see that

$$\begin{aligned} F((\iota \otimes \omega_{p,q})(V)^*) &= (\iota \otimes \omega_{J\nabla^{-\frac{1}{2}}Jq,\,J\nabla^{\frac{1}{2}}Jp})(V) \\ &= (\iota \otimes \omega_{Jp,Jq})(V^*) = (\iota \otimes \omega_{Jq,Jp})(V)^* = F((\iota \otimes \omega_{p,q})(V))^* \ . \end{aligned}$$

The result follows by linearity. ■

Therefore, proposition 3.13 justifies the following definition :

Definition 9.12 *We define R_u to be the anti-*-homomorphism from A_u into A_u such that $R_u((\iota \otimes \omega_{p,q})(V)) = (\iota \otimes \omega_{Jq,Jp})(V)$ for every $p, q \in \Lambda(A)$.*

Corollary 9.13 *We have that $(R_u \otimes \iota)(V) = (\iota \otimes (J.^*J))(V) = (\iota \otimes \hat{R}_r)(V)$.*

Corollary 9.14 *We have for every $\omega \in B_0(H)^*$ that $R_u((\iota \otimes \omega)(V)) = (\iota \otimes \omega(J.^*J))(V)$.*

Using this corollary, it is easy to see that the following holds.

Proposition 9.15 *The mapping R_u is an involutive anti-*-automorphism on A_u.*

Thanks to corollary 9.13, the antimultiplicativity of $\hat{R}_r$ gives us the following commutation relation.

Result 9.16 *The equality $\chi(R_u \otimes R_u)\Delta_u = \Delta_u \, R_u$ holds.*

Proof : By corollary 9.13 and proposition 4.12, we have that

$$\begin{aligned}(\Delta_u \, R_u \otimes \iota)(V) &= (\Delta_u \otimes \hat{R}_r)(V) = (\iota \otimes \iota \otimes \hat{R}_r)((\Delta_u \otimes \iota)(V)) \\ &= (\iota \otimes \iota \otimes \hat{R}_r)(V_{13}V_{23}) = (\iota \otimes \iota \otimes \hat{R}_r)((\chi \otimes \iota)(V_{23}V_{13})) \\ &= (\chi \otimes \iota)((\iota \otimes \iota \otimes \hat{R}_r)(V_{23}V_{13})) \ .\end{aligned}$$

Because $\hat{R}_r$ is antimultiplicative, this equation implies that

$$\begin{aligned}(\Delta_u \, R_u \otimes \iota)(V) &= (\chi \otimes \iota)\big((\iota \otimes \iota \otimes \hat{R}_r)(V_{13}) \, (\iota \otimes \iota \otimes \hat{R}_r)(V_{23})\big) \\ &= (\chi \otimes \iota)(\, (R_u \otimes \iota \otimes \iota)(V_{13}) \, (\iota \otimes R_u \otimes \iota)(V_{23}) \,) = (\chi \otimes \iota)((R_u \otimes R_u \otimes \iota)(V_{13}V_{23})) \\ &= (\chi(R_u \otimes R_u) \otimes \iota)((\Delta_u \otimes \iota)(V)) = (\chi(R_u \otimes R_u)\Delta_u \otimes \iota)(V) \ .\end{aligned}$$

Consequently, we get for every $\omega \in B_0(H)^*$ that $\Delta_u\big(R_u((\iota \otimes \omega)(V))\big) = \chi\big(\,(R_u \otimes R_u)(\,\Delta_u((\iota \otimes \omega)(V))\,)\,\big)$. The result follows. ■

It is not difficult to prove that R_u and τ_u commute :

Result 9.17 *We have for every $t \in \mathbb{R}$ that $(\tau_u)_t \, R_u = R_u \, (\tau_u)_t$.*

Proof : Choose $p, q \in H$. We know that ∇^{it} and J commute. Therefore,

$$\begin{aligned}R_u\big((\tau_u)_t((\iota \otimes \omega_{p,q})(V))\big) &= R_u((\iota \otimes \omega_{\nabla^{it}p,\nabla^{it}q})(V)) \\ &= (\iota \otimes \omega_{J\nabla^{it}q,J\nabla^{it}p})(V) = (\iota \otimes \omega_{\nabla^{it}Jq,\nabla^{it}Jp})(V) \\ &= (\tau_u)_t((\iota \otimes \omega_{Jq,Jp})(V)) = (\tau_u)_t\big(R_u((\iota \otimes \omega_{p,q})(V))\big) \ .\end{aligned}$$

The result follows. ■

At the end of this section, we prove the polar decomposition of the antipode on A_u.

Theorem 9.18 *We have for every $a \in A$ that $R_u\big((\tau_u)_{-\frac{i}{2}}(\pi_u(a))\big) = \pi_u(S(a))$.*

Proof : Choose $b, c \in A$ and put $d = (\iota \odot \varphi)(\Delta(c^*)(1 \otimes b))$. As usual, we have that $\pi_u(d) = (\iota \otimes \omega_{\Lambda(b),\Lambda(c)})(V)$. By result 9.3, this implies that

$$\begin{aligned}(\tau_u)_{-\frac{i}{2}}(\pi_u(d)) &= (\iota \otimes \omega_{\nabla^{i(-\frac{i}{2})}\Lambda(b),\nabla^{i(\overline{-\frac{i}{2}})}\Lambda(c)})(V) \\ &= (\iota \otimes \omega_{\nabla^{\frac{1}{2}}\Lambda(b),\nabla^{-\frac{1}{2}}\Lambda(c)})(V)\end{aligned}$$

Hence, using corollary 9.14, we get that

$$\begin{aligned}R_u\big((\tau_u)_{-\frac{i}{2}}(\pi_u(d))\big) &= (\iota \otimes \omega_{J\nabla^{-\frac{1}{2}}\Lambda(c),J\nabla^{\frac{1}{2}}\Lambda(b)})(V) \\ &= (\iota \otimes \omega_{T^*\Lambda(c),T\Lambda(b)})(V) = (\iota \otimes \omega_{\Lambda(\rho(c^*)),\Lambda(b^*)})(V)\ ,\end{aligned}$$

where we used the remarks after proposition 2.15 in the last equality. Using lemma 4.3 once again, we see that

$$R_u\big((\tau_u)_{-\frac{i}{2}}(\pi_u(d))\big) = \pi_u\big(\,(\iota \odot \varphi)(\,\Delta(b)(1 \otimes \rho(c^*))\,)\,\big)$$

Therefore, equation 1 implies that

$$R_u\big((\tau_u)_{-\frac{i}{2}}(\pi_u(d))\big) = \pi_u\big((\iota \odot \varphi)((1 \otimes c^*)\Delta(b))\big) = \pi_u\big(\,S(\,(\iota \odot \varphi)(\Delta(c^*)(1 \otimes b))\,)\,\big) = \pi_u(S(d))\ .$$

The theorem follows by linearity. ■

10 The left and right Haar weight on (A_u, Δ_u)

In this section, we introduce the left and right Haar weight on A_u. For this, we will use the left and right Haar weight on A_r and the bridge mapping π. Once we have our weights, the proofs of properties about these weights are completely analogous as the proofs of their reduced counterparts.

Definition 10.1 *We define the weight $\varphi_u = \varphi_r\,\pi$, then φ_u is a densely defined lower semi-continuous weight on A_u such that $\pi_u(A) \subseteq \mathcal{M}_{\varphi_u}$ and $\varphi_u(\pi_u(a)) = \varphi(a)$ for every $a \in A$.*

Although φ_r is a faithful weight on A_r, the weight φ_u does not have to be faithful. It will turn out that the quantum group (A_u, Δ_u) satisfies the definition of Masuda, Nakagami & Woronowicz except for the faithfulness of the left Haar weight.

Definition 10.2 *We use the GNS-construction (H, Λ_r, ι) for φ_r to define the GNS-construction for φ_u. We define the mapping Λ_u from $\mathcal{N}_{\varphi_u}$ into H such that $\Lambda_u(a) = \Lambda_r(\pi(a))$ for every $a \in \mathcal{N}_{\varphi_u}$. Then (H, Λ_u, π) is a GNS-construction for φ_u such that $\Lambda_u(\pi_u(a)) = \Lambda(a)$ for every $a \in A$.*

Proposition 10.3 *The weight φ_u is a KMS-weight on A_u with modular group σ_u.*

This proposition follows immediately from the fact that $\pi\,(\sigma_u)_t = (\sigma_r)_t\,\pi$ for every $t \in \mathbb{R}$ (remarks after definition 8.4).
Because φ_u is not faithful, the modular group is not uniquely determined. However, by imposing an extra condition, the modular group will be uniquely determined (see proposition 11.18).

In notation 4.6, we introduced the notation $\omega'_{v,w}$ for every $v, w \in H$. Then the proofs of the next two essential results are completely analogous as the proofs of proposition 6.9 and 6.10 of [9].

Proposition 10.4 *Let N be a dense left ideal in A_u such that $(\omega'_{v,v} \otimes \iota)\Delta_u(x)$ belongs to N for every $v \in H$ and $x \in N$. Then $\pi_u(A) \subseteq N$.*

Proposition 10.5 *Let N be a dense left ideal in A_u such that $(\iota \otimes \omega'_{v,v})\Delta_u(x)$ belongs to N for every $v \in H$ and $x \in N$. Then $\pi_u(A) \subseteq N$.*

Then we can prove the rather important result that φ_u is completely determined by its values on $\pi_u(A)$:

Theorem 10.6 *The set $\pi_u(A)$ is a core for Λ_u.*

The proof of this theorem is completely the analogous as the proof of theorem 6.12 of [9]. This starts with lemma 6.4 of [9] and goes on to theorem 6.12 of [9] itself.

Thanks to this result, we get the following strong form of left invariance (see the proof of theorem 6.13 of [9]). For used notations, we refer again to the appendix.

Theorem 10.7 *Consider $x \in \mathcal{M}_{\varphi_u}$. Then $\Delta_u(x)$ belongs to $\overline{\mathcal{M}}_{\iota\otimes\varphi_u}$ and $(\iota \otimes \varphi_u)\Delta(x) = \varphi_u(x)\,1$.*

This theorem justifies to call φ_u the left Haar weight of (A_u, Δ_u).

Then we get also the following weak form of left invariance.

Corollary 10.8 *Consider $x \in \mathcal{M}_{\varphi_u}$ and $\omega \in A_u^*$. Then $(\omega \otimes \iota)\Delta_u(x)$ belongs to $\mathcal{M}_{\varphi_u}$ and $\varphi_u((\omega \otimes \iota)\Delta_u(x)) = \varphi_u(x)\,\omega(1)$.*

Using the counit ε_u, the next result is trivial. To prove the analog of it in the reduced case, we had to do a lot more of work.

Proposition 10.9 *Let x be an element in A_u^+ such that $(\omega \otimes \iota)\Delta_u(x)$ belongs to $\mathcal{M}_{\varphi_u}^+$ for every $\omega \in (A_u)^*_+$. Then x belongs to $\mathcal{M}_{\varphi_u}^+$.*

Of course, the result on A_r (see theorem 3.11 of [12]) allows us to get a stronger version of this proposition.

Proposition 10.10 *Let x be an element in A_u^+ such that $(\omega'_{v,v} \otimes \iota)\Delta_u(x)$ belongs to $\mathcal{M}_{\varphi_u}^+$ for every $v \in H$. Then x belongs to $\mathcal{M}_{\varphi_u}^+$.*

In the same way as we prove proposition 6.15 of [9], we can prove the strong left invariance proposed in the definition of Masuda, Nakagami & Woronowicz.

Proposition 10.11 *Consider $a, b \in \mathcal{N}_{\varphi_u}$ and $\omega \in A_u^*$ such that $\omega R_u\,(\tau_u)_{-\frac{i}{2}}$ is bounded and call θ the unique element in A_u^* which extends $\omega R_u\,(\tau_u)_{-\frac{i}{2}}$. Then $b^*\,(\omega \otimes \iota)\Delta_u(a)$ and $(\theta \otimes \iota)(\Delta_u(b^*))\,a$ belong to $\mathcal{M}_{\varphi_u}$ and*

$$\varphi_u(\,b^*\,(\omega \otimes \iota)\Delta_u(a)\,) = \varphi_u(\,(\theta \otimes \iota)(\Delta_u(b^*))\,a\,)\ .$$

In definition 9.2 of [9], the right Haar weight ψ_r on (A_r, Δ_r) (which was denoted by ψ there) was defined by using the left Haar weight and the anti-unitary antipode R_r : $\psi_r = \varphi_r\,R_r$.

We used this weight ψ_r to define a non-zero positive right invariant linear ψ on the algebraic quantum group (A, Δ) (see definition 9.7 of [9]). We found that $\pi_r(A)$ is a subset of $\mathcal{M}_{\psi_r}$ and defined ψ in such a way that $\psi(a) = \psi_r(\pi_r(a))$ for every $a \in A$.

Definition 10.12 *We define the weight $\psi_u = \psi_r\,\pi$. So ψ_u is a densely defined lower semi-continuous weight on A_u such that $\psi_u(\pi_u(a)) = \psi(a)$ for every $a \in A$.*

Using the fact that $\psi_r = \varphi_r\,R$ and $\pi R_u = R_r\,\pi$, we get the following result :

Result 10.13 *The equality $\psi_u = \varphi_u\,R_u$ holds.*

This suggests the following definition :

Definition 10.14 *We define the norm-continuous one-parameter group σ'_u on A_u such that $(\sigma'_u)_t = R_u\,(\sigma_u)_{-t}\,R_u$ for every $t \in \mathbb{R}$.*

Corollary 10.15 *The weight ψ_u is a KMS-weight on A_u with modular group σ'_u.*

It is straightforward to check that $\pi\,(\sigma'_u)_t = (\sigma'_r)_t\,\pi$ for every $t \in \mathbb{R}$.

Combining results 9.4 and 9.16, we get immediately the following one.

Result 10.16 *We have for every $t \in \mathbb{R}$ that $((\sigma'_u)_t \otimes (\tau_u)_{-t})\Delta_u = \Delta_u\,(\sigma'_u)_t$*

In the same way as for φ_u, we can prove the following important result about ψ_u :

Theorem 10.17 *Let $(H_{\psi_u}, \Lambda_{\psi_u}, \pi_{\psi_u})$ be a GNS-construction for ψ_u. Then $\pi_u(A)$ is a core for Λ_{ψ_u}.*

As usual, the left invariance of φ_u is transferred to the right invariance of ψ_u :

Theorem 10.18 *Consider $x \in \mathcal{M}_{\psi_u}$. Then $\Delta_u(x)$ belongs to $\overline{\mathcal{M}}_{\psi_u \otimes \iota}$ and $(\psi_u \otimes \iota)\Delta_u(x) = \psi_u(x)\,1$.*

This theorem justifies to call ψ_u the right Haar weight of (A_u, Δ_u).

Corollary 10.19 *Consider $x \in \mathcal{M}_{\psi_u}$ and $\omega \in A_u^*$. Then $(\iota \otimes \omega)\Delta(x)$ belongs to $\mathcal{M}_{\psi_u}$ and $\psi_u((\iota \otimes \omega)\Delta_u(x)) = \psi_u(x)\,\omega(1)$.*

We will end this section with a natural formula for V in terms of the GNS-construction of φ_u. As before, we refer to the appendix for the used notations.

Proposition 10.20 *Consider $a \in A_u$ and $b \in \mathcal{N}_{\varphi_u}$. Then $\Delta(b)(a \otimes 1)$ belongs to $\mathcal{N}_{\iota \otimes \varphi_u}$ and $V(\iota \otimes \Lambda_u)(\Delta(b)(a \otimes 1)) = a \otimes \Lambda_u(b)$.*

Proof : First take a sequence $(a_n)_{n=1}^\infty$ in A such that $(\pi_u(a_n))_{n=1}^\infty$ converges to a. Because $\pi_u(A)$ is a core for Λ_u, we get the existence of a sequence $(b_n)_{n=1}^\infty$ in A such that $(\pi_u(b_n))_{n=1}^\infty$ converges to b and $\big(\Lambda_u(\pi_u(b_n))\big)_{n=1}^\infty$ converges to $\Lambda_u(b)$.
This implies immediately that the sequence

$$\big(\,\Delta_u(\pi_u(b_n))(\pi_u(a_n) \otimes 1)\,\big)_{n=1}^\infty \rightarrow \Delta_u(b)(a \otimes 1)\ .$$

Using the definition of V, we get for every $n \in \mathbb{N}$ that $\Delta_u(\pi_u(b_n))(\pi_u(a_n)\otimes 1) = (\pi_u\otimes\pi_u)(\Delta(b_n)(a_n\otimes 1)) \in \pi_u(A) \odot \pi_u(A) \subseteq D(\iota \otimes \Lambda_u)$ and

$$\begin{aligned}(\iota \otimes \Lambda_u)\big(\Delta_u(\pi_u(b_n))(\pi_u(a_n) \otimes 1)\big) &= (\iota \otimes \Lambda_u)\big((\pi_u \otimes \pi_u)(\Delta(b_n)(a_n \otimes 1))\big) \\ &= (\pi_u \odot \Lambda)(\Delta(b_n)(a_n \otimes 1)) = V^*\,(\pi_u(a_n) \otimes \Lambda(b_n)) = V^*\,\big(\pi_u(a_n) \otimes \Lambda_u(\pi_u(b_n))\big)\end{aligned}$$

This implies that the sequence

$$\big(\,(\iota\otimes\Lambda_u)(\,\Delta_u(\pi_u(b_n))(\pi_u(a_n)\otimes 1)\,)\,\big)_{n=1}^{\infty}\rightarrow V^*(a\otimes\Lambda_u(b))\ .$$

Hence, the closedness of $\iota\otimes\Lambda_u$ implies that $\Delta_u(b)(a\otimes 1)$ belongs to $D(\iota\otimes\Lambda_u)$ and

$$(\iota\otimes\Lambda_u)(\Delta_u(b)(a\otimes 1))=V^*\,(a\otimes\Lambda_u(b))\ .$$

■

11 Invariance properties of bi-C*-isomorphisms and group-like elements

In this section, we prove some relative invariance properties of bi-C*-isomorphism and group-like elements. The results are essentially the same as the results of section 7 of [9]. However, some proofs have to be altered because the left Haar weight on (A_u,Δ_u) is not necessarily faithful.

The proof of the following proposition is the same as the proof of proposition 7.1 of [9].

Proposition 11.1 *Consider *-automorphisms α and β on A_u such that $(\beta\otimes\alpha)\Delta_u=\Delta_u\,\alpha$. Then there exists a strictly positive number r such that $\varphi_u\,\alpha=r\,\varphi_u$.*

As before (see lemma 5.1), the mapping β above will automatically satisfy $(\beta\otimes\beta)\Delta_u=\Delta_u\,\beta$. Therefore,

Corollary 11.2 *Consider *-automorphisms α and β on A_u such that $(\beta\otimes\alpha)\Delta_u=\Delta_u\,\alpha$. Then there exist strictly positive numbers r and s such that $\varphi_u\,\alpha=r\,\varphi_u$ and $\varphi_u\,\beta=s\,\varphi_u$.*

In a similar way as in proposition 5.3, we get the following result :

Proposition 11.3 *Consider *-automorphisms α and β on A_u such that $(\alpha\otimes\beta)\Delta_u=\Delta_u\,\alpha$. Then there exists a strictly positive number r such that $\varphi_u\,\alpha=r\,\varphi_u$ and $\varphi_u\,\beta=r\,\varphi_u$.*

We want to prove a version of proposition 7.2 of [9] on the level of A_u. In the proof of proposition 7.2 of [9], we use that φ_r is faithful whereas φ_u does not have to be faithful. Therefore, we have to go around it in another way.

In section 5, we lifted certain *-automorphisms from the reduced to the universal level. In the next part of this section, we will consider the reverse process and show that both processes are each other inverse.

Therefore, consider *-automorphisms α,β on A_u such that $(\alpha\otimes\beta)\Delta_u=\Delta_u\,\alpha$. By proposition 11.3, there exists a strictly positive number r such that $\varphi_u\,\alpha=r\,\varphi_u$ and $\varphi_u\,\beta=r\,\varphi_u$.

Define unitary operators $u,v\in B(H)$ such that $u\Lambda_u(a)=r^{-\frac{1}{2}}\,\Lambda_u(\alpha(a))$ and $v\Lambda_u(a)=r^{-\frac{1}{2}}\,\Lambda_u(\beta(a))$. Then $\pi(\alpha(a))=u\pi(a)u^*$ and $\pi(\beta(a))=v\pi(a)v^*$ for every $a\in A$. This implies immediately that $uA_ru^*=A_r$ and $vA_rv^*=A_r$.

Definition 11.4 *Define the *-automorphisms α_r and β_r on A_r such that $\alpha_r(x)=uxu^*$ and $\beta_r(x)=vxv^*$ for every $x\in A_r$.*

Then $\pi\alpha = \alpha_r\,\pi$ and $\pi\beta = \beta_r\,\pi$. This implies immediately that $(\alpha_r \otimes \beta_r)\Delta_r = \Delta_r\,\alpha_r$. So we can use the results of section 5.

Because $\pi\alpha = \alpha_r\,\pi$, $\pi\beta = \beta_r\,\pi$ and $\varphi_u = \varphi_r\,\pi$, we get also that $\varphi_r\,\alpha_r = r\,\varphi_r$ and $\varphi_r\,\beta_r = r\,\varphi_r$. Using definition 10.2, we see also that $u\Lambda_r(a) = r^{-\frac{1}{2}}\,\Lambda_r(\alpha_r(a))$ and $v\Lambda_r(a) = r^{-\frac{1}{2}}\,\Lambda_r(\beta_r(a))$ for every $a \in \mathcal{N}_{\varphi_r}$.

With the notations used above, we get the following lemma.

Lemma 11.5 *We have that* $(\alpha \otimes \iota)(V) = (1 \otimes u^*)V(1 \otimes v)$ *and* $(\beta \otimes \iota)(V) = (1 \otimes v^*)V(1 \otimes v)$.

Proof : Choose $a, b \in A$ and $p, q \in \mathcal{N}_{\varphi_u}$. Then

$$\begin{aligned} \langle(\alpha \otimes \iota)(V)\,(a \otimes \Lambda_u(p)), b \otimes \Lambda_u(q)\rangle &= \alpha\big(\langle V(\alpha^{-1}(a) \otimes \Lambda_u(p)), \alpha^{-1}(b) \otimes \Lambda_u(q)\rangle\big) \\ &= \alpha\big(\langle \alpha^{-1}(a) \otimes \Lambda_u(p), V^*(\alpha^{-1}(b) \otimes \Lambda_u(q))\rangle\big)\ . \qquad \text{(a)} \end{aligned}$$

Using proposition 10.20, we get that $\Delta_u(q)(\alpha^{-1}(b) \otimes 1)$ belongs to $\mathcal{N}_{\iota\otimes\varphi_u}$ and $V^*(\alpha^{-1}(b) \otimes \Lambda_u(q)) = (\iota \otimes \Lambda_u)(\Delta_u(q)(\alpha^{-1}(b) \otimes 1))$.
So we see that $(\alpha^{-1}(b^*) \otimes 1)\Delta_u(q^*)(\alpha^{-1}(a) \otimes p)$ belongs to $\mathcal{M}_{\iota\otimes\varphi_u}$ and

$$\begin{aligned} (\iota \otimes \varphi_u)((\alpha^{-1}(b^*) \otimes 1)\Delta_u(q^*)(\alpha^{-1}(a) \otimes p)) &= \langle \alpha^{-1}(a) \otimes \Lambda_u(p), (\iota \otimes \Lambda_u)(\Delta_u(q)(\alpha^{-1}(b) \otimes 1))\rangle \\ &= \langle \alpha^{-1}(a) \otimes \Lambda_u(p), V^*(\alpha^{-1}(b) \otimes \Lambda_u(q))\rangle\ . \end{aligned}$$

Because $\varphi_u\,\beta = r\,\varphi_u$, this implies that $(\alpha \otimes \beta)((\alpha^{-1}(b^*) \otimes 1)\Delta_u(q^*)(\alpha^{-1}(a) \otimes p))$ belongs to $\mathcal{M}_{\iota\otimes\varphi_u}$ and

$$\begin{aligned} &(\iota \otimes \varphi_u)\big((\alpha \otimes \beta)((\alpha^{-1}(b^*) \otimes 1)\Delta_u(q^*)(\alpha^{-1}(a) \otimes p))\big) \\ &\quad = r\ \alpha\big((\iota \otimes \varphi_u)((\alpha^{-1}(b^*) \otimes 1)\Delta_u(q^*)(\alpha^{-1}(a) \otimes p))\big) \\ &\quad = r\ \alpha\big(\langle \alpha^{-1}(a) \otimes \Lambda_u(p), V^*(\alpha^{-1}(b) \otimes \Lambda_u(q))\rangle\big) \\ &\quad = r\ \langle(\alpha \otimes \iota)(V)\,(a \otimes \Lambda_u(p)), b \otimes \Lambda_u(q)\rangle \end{aligned}$$

where we used equation (a) in the last equality.

Therefore, the equality $(\alpha_u \otimes \beta_u)\Delta_u = \Delta_u\,\alpha_u$ implies that $(b^* \otimes 1)\Delta_u(\alpha(q)^*)(a \otimes \beta(p))$ belongs to $\mathcal{M}_{\iota\otimes\varphi_u}$ and

$$(\iota \otimes \varphi_u)((b^* \otimes 1)\Delta_u(\alpha(q)^*)(a \otimes \beta(p)) = r\ \langle(\alpha \otimes \iota)(V)\,(a \otimes \Lambda_u(p)), b \otimes \Lambda_u(q)\rangle\ . \qquad (b)$$

We have also that $\beta(p)$ and $\alpha(q)$ belong to $\mathcal{N}_{\varphi_u}$ and $\Lambda_u(\beta(p)) = r^{\frac{1}{2}}\,v\Lambda_u(p)$ and $\Lambda_u(\alpha(q)) = r^{\frac{1}{2}}\,u\Lambda_u(q)$. Therefore, proposition 10.20 implies that $\Delta_u(\alpha(q))(b \otimes 1)$ belongs to $\mathcal{N}_{\iota\otimes\varphi_u}$ and

$$(\iota \otimes \Lambda_u)(\Delta_u(\alpha(q))(b \otimes 1)) = V^*(b \otimes \Lambda_u(\alpha(q))) = r^{\frac{1}{2}}\,V^*(1 \otimes u)(b \otimes \Lambda_u(q))\ .$$

This implies that

$$\begin{aligned} &(\iota \otimes \varphi_u)((b^* \otimes 1)\Delta_u(\alpha(q)^*)(a \otimes \beta(p)) = \langle a \otimes \Lambda_u(\beta(p)), (\iota \otimes \Lambda_u)(\Delta_u(\alpha(q))(b \otimes 1))\rangle \\ &\quad = r\,\langle(1 \otimes v)(a \otimes \Lambda_u(p)), V^*(1 \otimes u)(b \otimes \Lambda_u(q))\rangle = r\,\langle(1 \otimes u^*)V(1 \otimes v)(a \otimes \Lambda_u(p)), b \otimes \Lambda_u(q)\rangle \end{aligned}$$

Comparing this with (b), we get that

$$\langle(\alpha \otimes \iota)(V)\,(a \otimes \Lambda_u(p)), b \otimes \Lambda_u(q)\rangle = \langle(1 \otimes u^*)V(1 \otimes v)\,(a \otimes \Lambda_u(p)), b \otimes \Lambda_u(q)\rangle\ .$$

So we see that $(\alpha \otimes \iota)(V) = (1 \otimes u^*)V(1 \otimes v)$. The other equality is proven in a similar way. ∎

Using this lemma and referring to results 5.16 and 5.20, we get the following proposition.

Proposition 11.6 *Consider *-automorphisms α,β on A_u such that $(\alpha \otimes \beta)\Delta_u = \Delta_u \, \alpha$. Then we have that $(\alpha_r)_u = \alpha$ and $(\beta_r)_u = \beta$.*

The next proposition is even easier to prove.

Proposition 11.7 *Consider *-automorphisms α,β on A_r such that $(\alpha \otimes \beta)\Delta_r = \Delta_r \, \alpha$. Then we have that $(\alpha_u)_r = \alpha$ and $(\beta_u)_r = \beta$.*

This follows immediately from the fact that $(\alpha_u)_r \, \pi = \pi \, \alpha_u = \alpha\pi$ (see result 5.22) and similarly for β.

We can now use the above results to prove some kind of uniqueness result.

Proposition 11.8 *Consider *-automorphisms $\alpha_1, \alpha_2, \beta_1, \beta_2$ on A_u such that $(\alpha_1 \otimes \beta_1)\Delta_u = \Delta_u \, \alpha_1$, $(\alpha_2 \otimes \beta_2)\Delta_u = \Delta_u \, \alpha_2$ and $\pi\alpha_1 = \pi\alpha_2$. Then $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$.*

Proof : By the results above, we have *-automorphisms $(\alpha_1)_r$, $(\alpha_2)_r$, $(\beta_1)_r$ and $(\beta_2)_r$ on A_r such that $(\alpha_1)_r \, \pi = \pi\alpha_1$, $(\alpha_2)_r \, \pi = \pi\alpha_2$, $(\beta_1)_r \, \pi = \pi\beta_1$ and $(\beta_2)_r \, \pi = \pi\beta_2$.
By assumption, we have that $(\alpha_1)_r \, \pi = \pi\alpha_1 = \pi\alpha_2 = (\alpha_2)_r \, \pi$, which implies that $(\alpha_1)_r = (\alpha_2)_r$.
So, proposition 11.6 implies that $\alpha_1 = ((\alpha_1)_r)_u = ((\alpha_2)_r)_u = \alpha_2$. Because

$$(\alpha_1 \otimes \beta_1)\Delta_u = \Delta_u \, \alpha_1 = \Delta_u\alpha_2 = (\alpha_2 \otimes \beta_2)\Delta_u = (\alpha_1 \otimes \beta_2)\Delta_u \ ,$$

we get also that $\beta_1 = \beta_2$. ■

Using the fact that $\chi(R_u \otimes R_u)\Delta_u = \Delta_u \, R_u$, we get also the following result.

Corollary 11.9 *Consider *-automorphisms $\alpha_1, \alpha_2, \beta_1, \beta_2$ on A_u such that $(\beta_1 \otimes \alpha_1)\Delta_u = \Delta_u \, \alpha_1$, $(\beta_2 \otimes \alpha_2)\Delta_u = \Delta_u \, \alpha_2$ and $\pi\alpha_1 = \pi\alpha_2$. Then $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$.*

We will now give a first application of these uniqueness results.

Proposition 11.10 *Consider *-automorphisms α and β on A_u such that $(\beta \otimes \alpha)\Delta_u = \alpha$. Then $\alpha \, (\sigma_u)_t = (\sigma_u)_t \, \alpha$, $\beta \, (\sigma_u)_t = (\sigma_u)_t \, \beta$ and $\beta \, (\tau_u)_t = (\tau_u)_t \, \beta$ for every $t \in \mathbb{R}$.*

Proof : Fix $t \in \mathbb{R}$. Then there exist *-automorphisms α', β' on A_r such that $\alpha' \, \pi = \pi \, \alpha$ and $\beta' \, \pi = \pi \, \beta$ (This follows from definition 11.4 and the use of the map R_u). Then $(\beta' \otimes \alpha')\Delta_r = \Delta_r \, \alpha'$.
From proposition 7.2 of [9], we get that $\alpha' \, (\sigma_r)_t = (\sigma_r)_t \, \alpha'$. So

$$\pi \, \alpha \, (\sigma_u)_t = \alpha' \, (\sigma_r)_t \, \pi = (\sigma_r)_t \, \alpha' \, \pi = \pi \, (\sigma_u)_t \, \alpha \ .$$

At the same time, we have that

$$(\beta \, (\tau_u)_t \otimes \alpha \, (\sigma_u)_t)\Delta_u = \Delta_u \, \alpha \, (\sigma_u)_t \qquad \text{and} \qquad ((\tau_u)_t \, \beta \otimes (\sigma_u)_t \, \alpha)\Delta_u = \Delta_u \, (\sigma_u)_t \, \alpha$$

Therefore, corollary 11.9 implies that $\alpha \, (\sigma_u)_t = (\sigma_u)_t \, \alpha$ and $\beta \, (\tau_u)_t = (\tau_u)_t \, \beta$.

We have also that $(\beta \otimes \beta)\Delta_u = \Delta_u \, \beta$. So we get in a similar way that $\beta \, (\sigma_u)_t = (\sigma_u)_t \, \beta$. ■

Proposition 11.11 *Consider a *-automorphism β on A_u such that $(\beta \otimes \beta)\Delta_u = \Delta_u \, \beta$. Then $\beta \, R_u = R_u \, \beta$.*

Proof : Again, we get the existence of a *-automorphism β_r on A_r such that $\beta_r \pi = \pi \beta$. Then $(\beta_r \otimes \beta_r)\Delta_r = \Delta_r \beta_r$, which implies that $\beta_r R_r = R_r \beta_r$ by proposition 7.4 of [9]. Hence $\pi R_u \beta R_u = R_r \beta_r R_r \pi = \beta_r \pi = \pi \beta$.
Because we also have that $(R_u \beta R_u \otimes R_u \beta R_u)\Delta_u = \Delta_u R_u \beta R_u$, we get that $R_u \beta R_u = \beta$ by proposition 11.8. ■

We want also to prove a version of proposition 7.5 of [9] on the level of A_u. Again, we have to find a way around the (possible) non-faithfulness of φ_u.

The prove of the next result is completely analogous as the proof of lemma 7.11 of [9].

Result 11.12 *Consider an element $x \in M(A_u)$ such that $\Delta_u(x) = x \otimes 1$ or $\Delta_u(x) = 1 \otimes x$. Then x will belong to $\mathbb{C}\,1$.*

Proposition 11.13 *Consider a unitary element $v \in M(A_u)$ such that $\Delta_u(v) = v \otimes v$. Then there exists a strictly positive number λ such that $(\sigma_u)_t(v) = \lambda^{it}\, v$ for every $t \in \mathbb{R}$. Moreover, $(\tau_u)_t(v) = v$ for every $t \in \mathbb{R}$.*

Proof : Choose $t \in \mathbb{R}$. Because $((\tau_u)_t \otimes (\tau_u)_t)\Delta_u = \Delta_u\,(\tau_u)_t$, we get that $\Delta_u((\tau_u)_t(v)) = (\tau_u)_t(v) \otimes (\tau_u)_t(v)$.
Because $(\pi \otimes \pi)\Delta_u = \Delta_r\,\pi$, we have that $\Delta_r(\pi(v)) = \pi(v) \otimes \pi(v)$. Hence, proposition 7.5 of [9] implies that $(\tau_r)_t(\pi(v)) = \pi(v)$. So we get that $\pi((\tau_u)_t(v)) = (\tau_r)_t(\pi(v)) = \pi(v)$. Hence, proposition 7.4 implies that $(\tau_u)_t(v) = v$.
So we get that

$$\begin{aligned}\Delta_u(v^*(\sigma_u)_t(v)) &= \Delta_u(v)^*\Delta_u((\sigma_u)_t(v)) = (v^* \otimes v^*)\,((\tau_u)_t \otimes (\sigma_u)_t)(\Delta_u(v)) \\ &= (v^* \otimes v^*)\,((\tau_u)_t(v) \otimes (\sigma_u)_t(v)) = (v^* \otimes v^*)\,(v \otimes (\sigma_u)_t(v)) = 1 \otimes v^*(\sigma_u)_t(v)\ .\end{aligned}$$

Therefore, the previous result implies the existence of a complex number λ_t such that $v^*(\sigma_u)_t(v) = \lambda_t\,1$, so $(\sigma_u)_t(v) = \lambda_t\,v$.

We get easily the existence of a strictly positive number λ such that $\lambda^{it} = \lambda_t$ for every $t \in \mathbb{R}$. ■

The proof of the next result is completely analogous as the proof of proposition 7.7 of [9].

Proposition 11.14 *Consider a unitary element v in $M(A_u)$ such that $\Delta_u(v) = v \otimes v$. Then there exists a unique unitary element $x \in M(A)$ such that $\pi_u(x\,a) = v\,\pi_u(a)$ and $\pi_u(a\,x) = \pi_u(a)\,v$ for every $a \in A$. We have moreover that $\Delta(x) = x \otimes x$.*

Using this proposition, the proof of the following result is also similar to the proof of proposition 7.8 of [9].

Proposition 11.15 *Consider a unitary element v in $M(A_u)$ such that $\Delta_u(v) = v \otimes v$. Then $R(v) = v^*$.*

Using these results about unitary elements, the proofs of the following two results can be copied from proposition 7.9 of [9] and 7.10 of [9].

Proposition 11.16 *Let α be a strictly positive element affiliated with A_u such that $\Delta_u(\alpha) = \alpha \otimes \alpha$. Then there exists a unique strictly positive number λ such that $(\sigma_u)_t(\alpha) = \lambda^t \alpha$ for every $t \in \mathbb{R}$. We have moreover that $(\tau_u)_t(\alpha) = \alpha$ for every $t \in \mathbb{R}$ and $R_u(\alpha) = \alpha^{-1}$.*

At the end of this section, we want to mention some interesting consequences of the uniqueness results in this section : proposition 11.8 and corollary 11.9.

The first one follows immediately from proposition 11.8 .

Proposition 11.17 *Consider $t \in \mathbb{R}$ and let α be a *-automorphism on A_u such that $(\alpha \otimes \alpha)\Delta_u = \Delta_u \, \alpha$ and $\pi \, \alpha = (\tau_r)_t \, \pi$. Then $(\tau_u)_t = \alpha$.*

The next consequence deals with the uniqueness of the modular group of φ_u under some extra condition.

Proposition 11.18 *Consider a norm continuous one-parameter group α on A_u such that φ_u is KMS with respect to α and such that for every $t \in \mathbb{R}$, there exists a *-automorphism β_t on A_u such that $(\beta_t \otimes \alpha_t)\Delta_u = \Delta_u \, \alpha_t$. Then we have that $\alpha = \sigma_u$.*

If φ_u is KMS with respect to α, we have that $\pi \, \alpha_t = (\sigma_r)_t \, \pi$ for every $t \in \mathbb{R}$ (see [10]). We know also that $\pi \, (\sigma_u)_t = (\sigma_r)_t \, \pi$ for every $t \in \mathbb{R}$. Then the proposition follows immediately from corollary 11.9 .

12 The modular function of the quantum group (A_u, Δ_u)

Consider the modular function δ_r of the quantum group (A_r, Δ_r) (see definition 8.2 of [9], where it was denoted by δ). Again, we want to transform this modular function of (A_r, Δ_r) to a modular function of (A_u, Δ_u). For this, we will use the results of section 7.

Let us fix $t \in \mathbb{R}$. Using proposition 8.6 of [9], we have that δ_r^{it} is a unitary element in $M(A_r)$ such that

$$\Delta_r(\delta_r^{it}) = \delta_r^{it} \otimes \delta_r^{it} \; .$$

By definition 7.1 and proposition 7.4 , we get a unique unitary element $(\delta_r^{it})_u$ in $M(A_u)$ such that

$$\pi((\delta_r^{it})_u) = \delta_r^{it} \qquad \text{and} \qquad \Delta_u((\delta_r^{it})_u) = (\delta_r^{it})_u \otimes (\delta_r^{it})_u \tag{5}$$

Using corollary 7.6, we have moreover for every $\omega \in B_0(H)$ and $t \in \mathbb{R}$ that

$$(\iota \otimes \omega)(V) \, (\delta_r^{it})_u = (\iota \otimes \delta_r^{-it} \omega \delta_r^{it})(V) \tag{6}$$

From this, we get that as usual the following results :

1. We have for every $a \in A_u$ that the mapping $\mathbb{R} \to A_u : t \mapsto a \, (\delta_r^{it})_u$ is norm-continuous.
2. The mapping $\mathbb{R} \to M(A_u) : t \mapsto (\delta_r^{it})_u$ is a group homomorphism.

Therefore, the following definition is justified.

Definition 12.1 *We define δ_u to be the unique strictly positive element affiliated with A_u such that $\delta_u^{it} = (\delta_r^{it})_u$ for every $t \in \mathbb{R}$.*

Then δ_u is determined by the following two properties (see equation 5 and the remarks before it) :

Proposition 12.2 *We have that $\Delta_u(\delta_u) = \delta_u \otimes \delta_u$ and $\pi(\delta_u) = \delta_r$.*

Also the following holds by equation 6 :

Result 12.3 *We have for every $t \in \mathbb{R}$ and $\omega \in B_0(H)^*$ that $(\iota \otimes \omega)(V)\, \delta_u^{it} = (\iota \otimes \delta_r^{-it}\omega\delta_r^{it})(V)$.*

Corollary 12.4 *We have that $V(\delta_u \otimes \delta_r) = (1 \otimes \delta_r)V$.*

The next proposition is an immediate consequence of proposition 11.16.

Proposition 12.5 *We have for every $t \in \mathbb{R}$ that $(\tau_u)_t(\delta_u) = \delta_u$. Furthermore, $R_u(\delta_u) = \delta_u^{-1}$.*

In proposition 8.17 of [9], we found that $(\sigma_r)_t(\delta_r) = \nu^{-t}\,\delta_r$. As usual, we want to obtain this equation on the level of A_u.

Proposition 12.6 *We have for every $t \in \mathbb{R}$ that $(\sigma_u)_t(\delta_u) = \nu^{-t}\,\delta_u$.*

Proof : Choose $s \in \mathbb{R}$. Take $\omega \in B_0(H)^*$. Using result 12.3 and the formula after definition 8.4, we get that

$$\begin{aligned}(\sigma_u)_t((\iota \otimes \omega)(V))\,(\sigma_u)_t(\delta_u^{is}) &= (\sigma_u)_t((\iota \otimes \omega)(V)\,\delta_u^{is}) \\ &= (\sigma_u)_t((\iota \otimes \delta_r^{-is}\omega\delta_r^{is})(V)) = (\iota \otimes P^{it}\delta_r^{-is}\omega\delta_r^{is}\nabla^{-it})(V)\ .\end{aligned}$$

By proposition 8.17 of [9], we know that $(\sigma_r)_t(\delta_r^{is}) = \nu^{-ist}\,\delta_r^{is}$, which implies that $\nabla^{it}\delta_r^{is}\nabla^{-it} = \nu^{-ist}\,\delta_r^{is}$. By proposition 7.9 of [9], we have also that $(\tau_r)_{-t}(\delta_r^{-is}) = \delta_r^{-is}$, which by corollary 9.21 of [9] implies that $(K_r)_t(\delta_r^{-is}) = \delta_r^{-it}\,(\tau_r)_{-t}(\delta_r^{-is})\,\delta_r^{it} = \delta_r^{-is}$. Remembering that K_r is implemented by P, we get that $P^{it}\delta_r^{-is}P^{-it} = \delta_r^{-is}$.
So we see that

$$(\sigma_u)_t((\iota \otimes \omega)(V))\,(\sigma_u)_t(\delta_u^{is}) = \nu^{-ist}\,(\iota \otimes \delta_r^{-is}P^{it}\omega\nabla^{-it}\delta_r^{is})(V)\ .$$

Using result 12.3 and the formula after definition 8.4 once more, this implies that

$$(\sigma_u)_t((\iota \otimes \omega)(V))\,(\sigma_u)_t(\delta_u^{is}) = \nu^{-ist}\,(\iota \otimes P^{it}\omega\nabla^{-it})(V)\,\delta_u^{is} = \nu^{-ist}\,(\sigma_u)_t((\iota \otimes \omega)(V))\,\delta_u^{is}\ .$$

It follows that $(\sigma_u)_t(\delta_u^{is}) = \nu^{-ist}\,\delta_u^{is}$.

From this all, we get the desired equality. ∎

Corollary 12.7 *We have for every $t \in \mathbb{R}$ that $(\sigma'_u)_t(\delta_u) = \nu^{-t}\,\delta_u$.*

The uniqueness results (proposition 11.8 and corollary 11.9) will prove useful once again :

Result 12.8 *We have for every $a \in A_u$ and $t \in \mathbb{R}$ that $(\sigma'_u)_t(a) = \delta_u^{it}\,(\sigma_u)_t(a)\,\delta_u^{-it}$ and $(K_u)_t(a) = \delta_u^{-it}\,(\tau_u)_{-t}(a)\,\delta_u^{it}$.*

Proof : Take $t \in \mathbb{R}$ and define the *-automorphisms α and β on A_u such that $\alpha(a) = \delta_u^{it}\,(\sigma_u)_t(a)\,\delta_u^{-it}$ and $\beta(a) = \delta_u^{it}\,(K_u)_t(a)\,\delta_u^{-it}$ for every $a \in A_u$.
By proposition 9.9 of [9], we have for every $a \in A_u$ that

$$\pi(\alpha(a)) = \pi(\delta_u^{it}\,(\sigma_u)_t(a)\;\delta_u^{-it}) = \delta_r^{it}\,(\sigma_r)_t(\pi(a))\,\delta_r^{-it} = (\sigma'_r)_t(\pi(a)) = \pi((\sigma'_u)_t(a)) \; .$$

Using result 8.6, we have also for every $a \in A_u$ that

$$\Delta_u(\alpha(a)) = \Delta_u(\delta_u^{it}\,(\sigma_u)_t(a)\,\delta_u^{-it}) = (\delta_u^{it} \otimes \delta_u^{it})\,((\sigma_u)_t \otimes (K_u)_t)(\Delta_u(a))\,(\delta_u^{-it} \otimes \delta_u^{-it}) = (\alpha \otimes \beta)\Delta(a) \; .$$

On the other hand, we know by result 10.16 also that

$$\Delta_u\,(\sigma'_u)_t = ((\sigma'_u)_t \otimes (\tau_u)_{-t})\Delta_u \; .$$

Hence, proposition 11.8 implies that $(\sigma'_u)_t = \alpha$ and $(\tau_u)_{-t} = \beta$. The proposition follows. ■

Again, ψ_u is absolutely continuous with respect to φ_u and a Radon Nikodym derivative is given by δ_u.

First, we need some notations. Consider a C*-algebra B and a KMS-weight Υ on B with modular group Σ. Let α be a strictly positive element affiliated with C such that there exists a strictly positive number λ such that $\Sigma_t(\alpha) = \lambda^t\,\alpha$ for all $t \in \mathbb{R}$.
Then it is possible to define the KMS-weight $\Upsilon_\alpha = \Upsilon(\alpha^{\frac{1}{2}}\,.\,\alpha^{\frac{1}{2}})$. For a precise definition, see [10].

Proposition 12.9 *We have the equality* $\psi_u = (\varphi_u)_{\delta_u}$.

Proof : Remember that $\psi_u = \psi_r\,\pi$, $\varphi_u = \varphi_r\,\pi$ and $\pi(\delta_u) = \delta_r$.
By theorem 9.18 of [9], we know that $\psi_r = (\varphi_r)_{\delta_r}$. Hence,

$$(\varphi_u)_{\delta_u} = (\varphi_r\,\pi)_{\delta_u} = (\varphi_r)_{\pi(\delta_u)}\,\pi = (\varphi_r)_{\delta_r}\,\pi = \psi_r\,\pi = \psi_u \; .$$

■

Because φ_u does not have to be faithful, the Radon Nikodym derivative is not uniquely determined. But, as before, imposing an extra condition on the Radon Nikodym derivative implies the uniqueness.

Remember from proposition 11.16 that if α is a strictly positive element affiliated with A_r satisfying $\Delta_u(\alpha) = \alpha \otimes \alpha$, there exists a strictly positive number λ such that $(\sigma_u)_t(\alpha) = \lambda^t\,\alpha$ for every $t \in \mathbb{R}$ so that $(\varphi_u)_\alpha$ is defined.

Proposition 12.10 *Let α be a strictly positive element affiliated with A_u such that $\Delta_u(\alpha) = \alpha \otimes \alpha$ and $(\varphi_u)_\alpha = \psi_u$. Then α will be equal to δ_u.*

Proof : Becaue $(\varphi_u)_{\delta_u} = \psi_u = (\varphi_u)_\alpha$, we get that $\pi(\alpha) = \pi(\delta_u)$. Because moreover $\Delta_u(\alpha) = \alpha \otimes \alpha$ and $\Delta_u(\delta_u) = \delta_u \otimes \delta_u$, proposition 7.4 implies that $\alpha = \delta_u$. ■

In the next part of this section, we will prove the universal variant of proposition 8.12 of [9]. Remember from definition 8.13 of [9] that δ^z is defined on the algebraic level for every complex number z.

Proposition 12.11 *Consider $a \in A$ and $z \in \mathbb{C}$. Then $\pi_u(a)$ belongs to $\mathcal{D}(\delta_u^z)$ and $\delta_u^z\,\pi_u(a) = \pi_u(\delta^z\,a)$.*

Proof : Choose $b, c \in A$ and put $d = (\iota \odot \varphi)(\Delta(c^*)(1 \otimes b))$. Then $\pi_u(d) = (\iota \otimes \omega_{\Lambda(b),\Lambda(c)})(V)$. By result 12.3, we have for every $t \in \mathbb{R}$ that

$$\pi_u(d)\, \delta_u^{it} = (\iota \otimes \omega_{\delta_r^{-it}\Lambda(b),\delta_r^{-it}\Lambda(c)})(V) \ .$$

Choose $y \in \mathbb{C}$. By the previous equality, lemma 8.8 of [9] and definition 8.13 of [9], we get immediately that $\pi_u(d)\, \delta_u^{iy}$ is bounded and

$$\overline{\pi_u(d)\, \delta_u^{iy}} = (\iota \otimes \omega_{\delta^{-iy}\Lambda(b),\delta^{-i\,\overline{y}}\Lambda(c)})(V) = (\iota \otimes \omega_{\Lambda(\delta^{-iy}b),\Lambda(\delta^{-i\,\overline{y}}c)})(V)$$

Using lemma 4.3 once again, we get that

$$\begin{aligned} \overline{\pi_u(d)\, \delta_u^{iy}} &= \pi_u\big((\iota \odot \varphi)(\Delta(c^* \delta^{iy})(1 \otimes \delta^{-iy} b))\big) = \pi_u\big((\iota \odot \varphi)(\Delta(c^*)(\delta^{iy} \otimes \delta^{iy})(1 \otimes \delta^{-iy} b))\big) \\ &= \pi_u((\iota \odot \varphi)(\Delta(c^*)(1 \otimes b))\, \delta^{iy}) = \pi_u(d\, \delta^{iy}) \end{aligned}$$

This implies for every $p \in A$ and every $x \in \mathbb{C}$ that $\pi_u(a)\, \delta_u^x$ is bounded and $\overline{\pi_u(p)\, \delta_u^x} = \pi_u(p\, \delta^x)$.

So we find in particular that $\pi_u(a^*)\delta_u^{\overline{z}}$ is bounded and $\overline{\pi_u(a^*)\delta_u^{\overline{z}}} = \pi_u(a^* \delta^{\overline{z}})$. This implies that $\pi_u(a)$ belongs to $\mathcal{D}(\delta_u^z)$ and

$$\delta_u^z\, \pi_u(a) = \big(\overline{\pi_u(a^*)\, \delta_u^{\overline{z}}}\big)^* = \pi_u(a^* \delta^{\overline{z}})^* = \pi_u(\delta^z\, a) \ .$$

■

So we get in particular that $\delta_u^{it}\, \pi_u(A) \subseteq \pi_u(A)$ for every $t \in \mathbb{R}$. Using this fact we can prove the following result in the same way as proposition 8.14 of [9].

Proposition 12.12 *Consider $z \in \mathbb{C}$, then the set $\pi_u(A)$ is a core for δ_u^z.*

Using the theory of regular C*-valued weights (see [13]), we get an analytic version of the algebraic equality $(\varphi \odot \iota)\Delta(a) = \varphi(a)\, \delta$ where a is an element of A. (for used notations, see section 14.3).

The proof of the next proposition is the same as the proof of lemma 8.15 of [9].

Proposition 12.13 *Consider $a \in \mathcal{N}_{\varphi_u}$ and $b \in \mathcal{D}(\delta_u^{\frac{1}{2}})$. Then $\Delta(a)(1 \otimes b)$ belongs to $\mathcal{N}_{\varphi \otimes \iota}$ and*

$$\langle (\Lambda_u \otimes \iota)(\Delta(a)(1 \otimes b)), (\Lambda_u \otimes \iota)(\Delta(a)(1 \otimes b)) \rangle = \varphi_u(a^* a)\, (\delta_u^{\frac{1}{2}} b)^* (\delta_u^{\frac{1}{2}} b) \quad .$$

Theorem 12.14 *Let y be an element in $\mathcal{M}_{\varphi_u}^+$. Then $\Delta_u(y)$ belongs to $\hat{\mathcal{M}}_{\varphi_u \otimes \iota}$ and $(\varphi_u \otimes \iota)\Delta_u(y) = \varphi_u(y)\, \delta_u$.*

Proof : We refer to section 14.3 for used notations.
By definitions 14.16 and 14.17, the previous proposition implies for every $p \in \mathcal{N}_{\varphi_u}$ that $\Delta_u(p)$ belongs to $\tilde{\mathcal{N}}_{\varphi_u \otimes \iota}$, that $\mathcal{D}(\delta_u^{\frac{1}{2}}) \subseteq D\big((\Lambda_u \otimes \iota)(\Delta_u(p))\big)$ and that

$$\langle (\Lambda_u \otimes \iota)(\Delta_u(p))\, q, (\Lambda_u \otimes \iota)(\Delta_u(p))\, q \rangle = \varphi_u(p^* p)\, (\delta_u^{\frac{1}{2}} q)^* (\delta_u^{\frac{1}{2}} q) \qquad \text{(a)}$$

for every $q \in \mathcal{D}(\delta_u^{\frac{1}{2}})$.

Fix $c \in \mathcal{N}_{\varphi_u}$ for the rest of this proof. Using theorem 10.6, we get the existence of a sequence $(c_n)_{n=1}^\infty$ in A such that $(\pi_u(c_n))_{n=1}^\infty$ converges to c and $\big(\Lambda_u(\pi_u(c_n))\big)_{n=1}^\infty$ converges to $\Lambda_u(c)$.

Take $b \in A$. By the previous part, we know already that $\pi_u(b)$ is an element of $D\big((\Lambda_u \otimes \iota)(\Delta_u(c))\big)$.

Take $x \in D\big((\Lambda_u \otimes \iota)(\Delta_u(c))\big)$. Using definitions 14.16 and 14.17 once more, we see that $\Delta_u(c)(1 \otimes x)$ belongs to $\mathcal{N}_{\varphi_u \otimes \iota}$ and $(\Lambda_u \otimes \iota)(\Delta_u(c)(1 \otimes x)) = (\Lambda_u \otimes \iota)(\Delta_u(c))\, x$.

Take $a \in A$. Then we have also that $(\pi_u \odot \pi_u)(\Delta(a)(1 \otimes b))$ belongs to and $\mathcal{N}_{\varphi_u \otimes \iota}$ and

$$(\Lambda_u \otimes \iota)\big((\pi_u \odot \pi_u)(\Delta(a)(1 \otimes b))\big) = (\Lambda_u \otimes \iota)\big(\Delta_u(\pi_u(a))(1 \otimes \pi_u(b))\big) = (\Lambda_u \otimes \iota)\big(\Delta_u(\pi_u(a))\big)\, \pi_u(b) \ .$$

By section 1, there exist $e \in A$ such that $\Delta(a)(1 \otimes b) = \Delta(a)(1 \otimes b)(e \otimes 1)$.
We know that $\pi_u(e)$ belongs to $\mathcal{D}((\sigma_u)_i)$ and that $(\sigma_u)_i(\pi_u(e)) = \pi_u(\rho^{-1}(e))$.
This implies that

$$\begin{aligned}
&\langle (\Lambda_u \otimes \iota)\big(\Delta_u(\pi_u(a))\big)\, \pi_u(b), (\Lambda_u \otimes \iota)(\Delta_u(c))\, x \rangle \\
&\quad = \langle (\Lambda_u \otimes \iota)\big((\pi_u \odot \pi_u)(\Delta(a)(1 \otimes b))\big), (\Lambda_u \otimes \iota)(\Delta_u(c)(1 \otimes x)) \rangle \\
&\quad = (\varphi_u \otimes \iota)\big([\Delta_u(c)(1 \otimes x)]^* \, (\pi_u \odot \pi_u)(\Delta(a)(1 \otimes b))\big) \\
&\quad = (\varphi_u \otimes \iota)([\Delta_u(c)(1 \otimes x)]^* \, (\pi_u \odot \pi_u)(\Delta(a)(1 \otimes b))\, (\pi_u(e) \otimes 1)) \\
&\quad = (\varphi_u \otimes \iota)\big((\pi_u(\rho^{-1}(e)) \otimes 1)\, [\Delta_u(c)(1 \otimes x)]^* \, (\pi_u \odot \pi_u)(\Delta(a)(1 \otimes b))\big) \\
&\quad = (\varphi_u \otimes \iota)\big((\pi_u(\rho^{-1}(e)) \otimes x^*)\, \Delta_u(c^*)\, (\pi_u \odot \pi_u)(\Delta(a)(1 \otimes b))\big) \ . \qquad \text{(b)}
\end{aligned}$$

Notice that $\pi_u(\rho^{-1}(e))^* \otimes x$ and $(\pi_u \odot \pi_u)(\Delta(a)(1 \otimes b))$ belong to $\mathcal{N}_{\varphi_u \otimes \iota}$. This implies that the sequence

$$\big(\, (\varphi_u \otimes \iota)(\, (\pi_u(\rho^{-1}(e)) \otimes x^*)\, \Delta_u(\pi_u(c_n)^*)\, (\pi_u \odot \pi_u)(\Delta(a)(1 \otimes b))\,)\, \big)_{n=1}^{\infty}$$

converges to

$$(\varphi_u \otimes \iota)(\, (\pi_u(\rho^{-1}(e)) \otimes x^*)\, \Delta_u(c^*)\, (\pi_u \odot \pi_u)(\Delta(a)(1 \otimes b))\,) \ . \qquad \text{(c)}$$

We have also for every $n \in \mathbb{N}$ that $(\pi_u \odot \pi_u)((\rho^{-1}(e) \otimes 1)\Delta(c_n^* a)(1 \otimes b))$ belongs to $\mathcal{M}_{\varphi_u \otimes \iota}$ which by result 14.12 implies that

$$\begin{aligned}
&(\varphi_u \otimes \iota)(\, (\pi_u(\rho^{-1}(e)) \otimes x^*)\, \Delta_u(\pi_u(c_n)^*)\, (\pi_u \odot \pi_u)(\Delta(a)(1 \otimes b))\,) \\
&\quad = (\varphi_u \otimes \iota)(\, (1 \otimes x^*)(\pi_u \odot \pi_u)((\rho^{-1}(e) \otimes 1)\Delta(c_n^* a)(1 \otimes b))\,) \\
&\quad = x^* \ (\varphi_u \otimes \iota)(\, (\pi_u \odot \pi_u)((\rho^{-1}(e) \otimes 1)\Delta(c_n^* a)(1 \otimes b))\,) \\
&\quad = x^* \ (\varphi \odot \pi_u)((\rho^{-1}(e) \otimes 1)\Delta(c_n^* a)(1 \otimes b)) = x^* \ (\varphi \odot \pi_u)(\Delta(c_n^* a)(1 \otimes b)(e \otimes 1)) \\
&\quad = x^* \ (\varphi \odot \pi_u)(\Delta(c_n^* a)(1 \otimes b)) = \varphi(c_n^* a)\ x^* \pi_u(\delta\, b) = \langle \Lambda_u(\pi_u(a)), \Lambda_u(\pi_u(c_n)) \rangle\ x^* \pi_u(\delta\, b)
\end{aligned}$$

were the algebraic version was used in the second last equality. This implies that the sequence

$$\big(\, (\varphi_u \otimes \iota)(\, (\pi_u(\rho^{-1}(e)) \otimes x^*)\, \Delta_u(\pi_u(c_n)^*)\, (\pi_u \odot \pi_u)(\Delta(a)(1 \otimes b))\,)\, \big)_{n=1}^{\infty}$$

converges to $\langle \Lambda_u(\pi_u(a)), \Lambda_u(c) \rangle\ x^* \pi_u(\delta\, b)$. Combining this with (c), we see that

$$(\varphi_u \otimes \iota)\big((\pi_u(\rho^{-1}(e)) \otimes x^*)\, \Delta_u(c^*)\, (\pi_u \odot \pi_u)(\Delta(a)(1 \otimes b))\big) = \langle \Lambda_u(\pi_u(a)), \Lambda_u(c) \rangle\ x^* \pi_u(\delta\, b) \ .$$

Hence, equality (b) implies that

$$\langle (\Lambda_u \otimes \iota)\big(\Delta_u(\pi_u(a))\big)\, \pi_u(d), (\Lambda_u \otimes \iota)(\Delta_u(c))\, x \rangle = \langle \Lambda_u(\pi_u(a)), \Lambda_u(c) \rangle\ x^* \pi_u(\delta\, b) \ .$$

From this last equality, we get for every $n \in \mathbb{N}$ that

$$\langle (\Lambda_u \otimes \iota)\big(\Delta_u(\pi_u(c_n))\big)\, \pi_u(b), (\Lambda_u \otimes \iota)(\Delta_u(c))\, x \rangle = \langle \Lambda_u(\pi_u(c_n)), \Lambda_u(c) \rangle\ x^* \pi_u(\delta\, d) \ .$$

We have chosen the sequence $(c_n)_{n=1}^\infty$ in such a way that $\bigl(\Lambda_u(\pi_u(c_n))\bigr)_{n=1}^\infty$ converges to $\Lambda_u(c)$.
By equality (a), we have for every $n \in \mathbb{N}$ that

$$\|(\Lambda_u \otimes \iota)\bigl(\Delta_u(\pi_u(c_n))\bigr)\,\pi_u(b) - (\Lambda_u \otimes \iota)\bigl(\Delta_u(c)\bigr)\,\pi_u(b)\|^2 = \|\Lambda_u(\pi_u(c_n)) - \Lambda_u(c)\|^2\ \|\delta_u^{\frac{1}{2}}\,\pi_u(b)\|^2\ .$$

This implies that the sequence $\Bigl(\,(\Lambda_u \otimes \iota)\bigl(\Delta_u(\pi_u(c_n))\bigr)\,\pi_u(d)\,\Bigr)_{n=1}^\infty$ converges to $(\Lambda_u \otimes \iota)(\Delta_u(c))\,\pi_u(d)$.
So we can conclude that

$$\langle(\Lambda_u \otimes \iota)(\Delta_u(c))\,\pi_u(b), (\Lambda_u \otimes \iota)(\Delta_u(c))\,x\rangle = \langle\Lambda_u(c), \Lambda_u(c)\rangle\ x^*\pi_u(\delta\,b)\ .$$

This last equality implies immediately that $(\Lambda_u \otimes \iota)(\Delta_u(c))\,\pi_u(b)$ belongs to $D((\Lambda_u \otimes \iota)(\Delta_u(c))^*)$ and that

$$(\Lambda_u \otimes \iota)(\Delta_u(c))^*\bigl((\Lambda_u \otimes \iota)(\Delta_u(c))\,\pi_u(b)\bigr) = \langle\Lambda_u(c), \Lambda_u(c)\rangle\ \pi_u(\delta\,b) = \varphi_u(c^*c)\ \delta_u\,\pi_u(b)\ .$$

So we see that $\pi_u(b)$ belongs to $D\bigl((\Lambda_u \otimes \iota)(\Delta_u(c))^*(\Lambda_u \otimes \iota)(\Delta_u(c))\bigr)$ and

$$(\Lambda_u \otimes \iota)(\Delta_u(c))^*(\Lambda_u \otimes \iota)(\Delta_u(c))\,\pi_u(b) = \varphi_u(c^*c)\ \delta_u\,\pi_u(b) \qquad \text{(d)}$$

Because δ_u is affiliated to A_u, we know that $1 + \varphi_u(c^*c)\,\delta_u$ has dense range in A_u. Because $\pi_u(A)$ is a core for δ_u, this implies that $(1 + \varphi_u(c^*c)\,\delta_u)\,\pi_u(A)$ is dense in A_u.
Hence, equation (d) implies that $\bigl(1 + (\Lambda_u \otimes \iota)(\Delta_u(c))^*(\Lambda_u \otimes \iota)(\Delta_u(c))\bigr)\,\pi_u(A)$ is dense in A_u. By the remarks after definition 14.17 and definition 14.18, this implies that $(\Lambda_u \otimes \iota)(\Delta_u(c))$ is regular so $\Delta_u(c)$ belongs to $\hat{\mathcal{N}}_{\varphi_u \otimes \iota}$
Because of equation (d) and the fact that $\pi_u(A)$ is a core of δ_u, we see that

$$\varphi_u(c^*c)\ \delta_u \subseteq (\Lambda_u \otimes \iota)(\Delta_u(c))^*(\Lambda_u \otimes \iota)(\Delta_u(c))\ .$$

But both are selfadjoint, so they must be equal. By definition 14.19, we get that

$$(\varphi_u \otimes \iota)(\Delta_u(c^*c)) = (\Lambda_u \otimes \iota)(\Delta_u(c))^*(\Lambda_u \otimes \iota)(\Delta_u(c)) = \varphi_u(c^*c)\ \delta_u\ .$$

■

13 The universal corepresentation of (A_u, Δ_u)

Remember from section 1 that we constructed an (algebraic) universal corepresentation $X \in M(A \odot \hat{A})$ for (A, Δ) in [11]. This element served as a link between corepresentations of (A, Δ) and *-homomorphisms on $\hat{A}$.

In this section, we will use this algebraic universal corepresentation to define a C*-algebraic universal corepresentation of (A_u, Δ_u).

It is clear that everything we did for A, we can also do for $\hat{A}$. So we will get a universal C*-algebraic quantum group $(\hat{A}_u, \hat{\Delta}_u)$ in the same way as we got the universal C*-algebraic quantum group (A_u, Δ_u). We will denote the canonical embedding from $\hat{A}$ into $\hat{A}_u$ by $\hat{\pi}_u$. We also define $\hat{\pi}$ as the unique *-homomorphism from $\hat{A}_u$ into $\hat{A}_r$ such that $\hat{\pi} \circ \hat{\pi}_u = \hat{\pi}_r$.

Remembering that X is a unitary element in $M(A \odot \hat{A})$, we can give give the following definition.

Definition 13.1 *We define the unitary element U in $M(A_u \otimes \hat{A}_u)$ in such a way that*

$$U\,(\pi_u \odot \hat{\pi}_u)(x) = (\pi_u \odot \hat{\pi}_u)(X\,x) \qquad \textit{and} \qquad (\pi_u \odot \hat{\pi}_u)(x)\,U = (\pi_u \odot \hat{\pi}_u)(x\,X)$$

for every $x \in A \odot \hat{A}$.

Notice that we also have that $(\Delta \odot \iota)(X) = X_{13}\, X_{23}$ and $(\iota \odot \hat{\Delta})(X) = X_{12}\, X_{23}$ which implies the following result :

Proposition 13.2 *We have that* $(\Delta \otimes \iota)(U) = U_{13}\, U_{23}$ *and* $(\iota \otimes \hat{\Delta})(U) = U_{12}\, U_{23}$.

The definition of U together with proposition 2.12 and proposition 4.9 will also immediately imply that $(\pi \odot \hat{\pi})(U) = W$ and $(\iota \otimes \hat{\pi})(U) = V$.

Consider $v, w \in H$.
In notation 4.6, we defined $\omega'_{v,w} \in A_u^*$ such that $\omega'_{v,w}(x) = \langle \pi(x)\, v, w \rangle$ for every $x \in A_u$.
In a similar way, we define $\hat{\omega}'_{v,w} \in \hat{A}_u^*$ such that $\hat{\omega}'_{v,w}(x) = \langle \hat{\pi}(x)\, v, w \rangle$ for every $x \in \hat{A}_u$.

Then we have the following equalities :

Result 13.3 *Consider $a, b \in A$, then the following two equalities hold :*

1. $(\iota \otimes \hat{\omega}'_{\Lambda(a),\Lambda(b)})(U) = \pi_u\big((\iota \odot \varphi)(\Delta(b^*)(1 \otimes a))\big)$
2. $(\omega'_{\Lambda(a),\Lambda(b)} \otimes \iota)(U) = \hat{\pi}_u(a \varphi b^*)$

Proof :

1. We get immediately that $\hat{\omega}'_{\Lambda(a),\Lambda(b)} = \omega_{\Lambda(a),\Lambda(b)}\, \hat{\pi}$. So, using the fact that $(\iota \otimes \hat{\pi})(U) = V$ and lemma 4.3, we get that
$$\begin{aligned}(\iota \otimes \hat{\omega}'_{\Lambda(a),\Lambda(b)})(U) &= (\iota \otimes \omega_{\Lambda(a),\Lambda(b)})\big((\iota \otimes \hat{\pi})(U)\big) \\ &= (\iota \otimes \omega_{\Lambda(a),\Lambda(b)})(V) = \pi_u\big((\iota \odot \varphi)(\Delta(b^*)(1 \otimes a))\big)\ .\end{aligned}$$

2. Choose $c, d \in A$. Take $\theta \in \hat{A}$. Then
$$\begin{aligned}(\omega'_{\Lambda(cd),\Lambda(b)} \otimes \iota)(U)\, \hat{\pi}_u(\theta) &= (\omega'_{\pi_r(c)\Lambda(d),\Lambda(b)} \otimes \iota)(U)\, \hat{\pi}_u(\theta) \\ &= (\omega'_{\Lambda(d),\Lambda(b)} \otimes \iota)\big(U(\pi_u(c) \otimes \hat{\pi}_u(\theta))\big)\ .\end{aligned}$$
Hence, the definition of U implies that
$$\begin{aligned}(\omega'_{\Lambda(cd),\Lambda(b)} \otimes \iota)(U)\, \hat{\pi}_u(\theta) &= (\omega'_{\Lambda(d),\Lambda(b)} \otimes \iota)\big((\pi_u \odot \hat{\pi}_u)(X(c \otimes \theta))\big) \\ &= \pi_u\big((d\varphi b^* \odot \iota)(X(c \otimes \theta))\big) \qquad (*)\end{aligned}$$
By result 6.8 of [11], we know that $(cd\varphi b^* \otimes \iota)(X) = cd\varphi b^*$. Looking at remark 4.4 of [11], this means that
$$(d\varphi b^* \odot \iota)(X(c \otimes \theta)) = (cd\varphi b^* \odot \iota)(X)\, \theta = (cd\varphi b^*)\, \theta$$
Substituting this in equation (*), we get that
$$(\omega'_{\Lambda(cd),\Lambda(b)} \otimes \iota)(U)\, \hat{\pi}_u(\theta) = \pi_u\big((cd\varphi b^*)\, \theta\big) = \pi_u(cd\varphi b^*)\, \pi_u(\theta)\ .$$
So we see that
$$(\omega'_{\Lambda cd,\Lambda(b)} \otimes \iota)(U) = \pi_u(cd\varphi b^*)\ .$$
Now the equality follows because $A^2 = A$.

■

Corollary 13.4 *We have the following two properties :*

1. *We have that A_u is a subset of the closure of $\{\, (\iota \otimes \omega)(U) \mid \omega \in \hat{A}_u^* \,\}$ in $M(A_u)$.*
2. *We have that $\hat{A}_u$ is a subset of the closure of $\{\, (\omega \otimes \iota)(U) \mid \omega \in A_u^* \,\}$ in $M(\hat{A}_u)$.*

Now we can prove easily the universality property of U (this kind of universality was introduced for compact quantum groups in [17]).

Theorem 13.5 *Consider a C^*-algebra C and a unitary corepresentation $\mathcal{U}$ of (A_u, Δ_u) on C. Then there exists a unique non-degenerate*-homomorphism θ from $\hat{A}_u$ into $M(C)$ such that $(\iota \otimes \theta)(U) = \mathcal{U}$.*

Proof : Uniqueness follows immediately from the previous corollary. We turn to the existence.

In proposition 7.10, we got a dense *-algebra $C_{\mathcal{U}}$ of C such that $\mathcal{U}\,(\pi_u(A) \odot C_{\mathcal{U}}) = \pi_u(A) \odot C_{\mathcal{U}}$ and $(\pi_u(A) \odot C_{\mathcal{U}})\,\mathcal{U} = \pi_u(A) \odot C_{\mathcal{U}}$.
We defined moreover in definition 7.11 the element $\hat{\mathcal{U}} \in M(A \odot C_{\mathcal{U}})$ such that $(\pi_u \odot \iota)(x)\,\mathcal{U} = (\pi_u \odot \iota)(x\,\hat{\mathcal{U}})$ and $\mathcal{U}\,(\pi_u \odot \iota)(x) = (\pi_u \odot \iota)(\hat{\mathcal{U}}\,x)$ for every $x \in A \odot C_{\mathcal{U}}$. Then $\hat{\mathcal{U}}$ is a unitary corepresentation of (A, Δ) on $C_{\mathcal{U}}$.

In proposition 6.11 of [11], we proved the existence of an algebraically non-degenerate *-homomorphism η from $\hat{A}$ into $M(C_{\mathcal{U}})$ such that $(\iota \odot \eta)(X) = \hat{\mathcal{U}}$.
Combining lemma 3.1 and the universality property of $\hat{A}_u$, we get the existence of a unique non-degenerate *-homomorphism θ from $\hat{A}_u$ into $M(C)$ such that $\theta(\pi_u(\omega))\,c = \eta(\omega)\,c$ and $c\,\theta(\pi_u(\omega)) = c\,\eta(\omega)$ for every $c \in C_{\mathcal{U}}$ and $\omega \in \hat{A}$.

Choose $a \in A$, $\omega \in \hat{A}$ and $c \in C$. Then

$$\begin{aligned}(\iota \otimes \theta)(U)\,(\pi_u(a) \otimes \theta(\pi_u(\omega))\,c\,) &= (\iota \otimes \theta)\big(U(\pi_u(a) \otimes \pi_u(\omega))\big)\,(1 \otimes c) \\ &= (\iota \otimes \theta)\big((\pi_u \odot \hat{\pi}_u)(X(a \otimes \omega))\big)\,(1 \otimes c)\end{aligned}$$

where we used the definition of U in the last equality. So we get that

$$(\iota \otimes \theta)(U)\,(\pi_u(a) \otimes \theta(\pi_u(\omega))\,c\,) = (\pi_u \odot \eta)(X(a \otimes \omega))\,(1 \otimes c)$$

If we use the fact that $(\iota \odot \eta)(X) = \hat{\mathcal{U}}$, this implies that

$$(\iota \otimes \theta)(U)\,(\pi_u(a) \otimes \theta(\pi_u(\omega))\,c\,) = (\pi_u \odot \iota)(\hat{\mathcal{U}}(a \otimes \eta(\omega)\,c\,)) = \mathcal{U}\,(\pi_u(a) \otimes \eta(\omega)\,c\,)$$

where we used the definition of $\hat{\mathcal{U}}$ in the last equality. Using the fact that $\eta(\hat{A})\,C_{\mathcal{U}} = C_{\mathcal{U}}$ (which is dense in C), we get that

$$(\iota \otimes \theta)(U) = \mathcal{U} \;.$$

■

So the previous theorem guarantees that the mapping which sends a non-degenerate *-homomorphism $\theta : \hat{A}_u \rightarrow M(C)$ into the corepresentation $(\iota \otimes \theta)(U)$, is a bijection between the set of non-degenerate *-homomorphisms from $\hat{A}_u$ into $M(C)$ and the set of unitary corepresentations of (A_u, Δ_u) on C.

In [23], A. Van Daele proved that we can identify (A, Δ) and $(\hat{\hat{A}}, \hat{\hat{\Delta}})$. In [11], we used this identification to prove that $\chi(X)$ is the universal algebraic corepresentation of $(\hat{A}, \hat{\Delta})$.
This will imply that $\chi(U)$ can be used in the same way to get a bijection from the set of non-degenerate *-homomorphisms on A_u and the set of unitary corepresentations of $(\hat{A}_u, \hat{\Delta}_u)$.

14 Appendix: some information about weights

In this section, we will collect some necessary information and conventions about weights and slicing with weights.

14.1 Weights

In this first section, we give some information about weights. The standard reference for lower semi-continuous weights is [3]. We start of with some standard notions concerning lower semi-continuous weights.

Consider a C^*-algebra A and a densely defined lower semi- continuous weight φ on A. We will use the following notations:

- $\mathcal{M}_\varphi^+ = \{\, a \in A^+ \mid \varphi(a) < \infty \,\}$
- $\mathcal{N}_\varphi = \{\, a \in A \mid \varphi(a^* a) < \infty \,\}$
- $\mathcal{M}_\varphi = \text{span}\, \mathcal{M}_\varphi^+ = \mathcal{N}_\varphi^* \mathcal{N}_\varphi$.

A GNS-construction of φ is by definition a triple $(H_\varphi, \pi_\varphi, \Lambda_\varphi)$ such that

- H_φ is a Hilbert space
- Λ_φ is a linear map from $\mathcal{N}_\varphi$ into H_φ such that
  1. $\Lambda_\varphi(\mathcal{N}_\varphi)$ is dense in H_φ
  2. We have for every $a, b \in \mathcal{N}_\varphi$, that $\langle \Lambda_\varphi(a), \Lambda_\varphi(b) \rangle = \varphi(b^* a)$

  Because φ is lower semi-continuous, Λ_φ is closed.
- π_φ is a non-degenerate representation of A on H_φ such that $\pi_\varphi(a)\, \Lambda_\varphi(b) = \Lambda_\varphi(ab)$ for every $a \in M(A)$ and $b \in \mathcal{N}_\varphi$. (The non-degeneracy of π_φ is a consequence of the lower semi-continuity of φ.)

The following concepts play a central role in the theory of lower semi-continuous weights.

Definition 14.1 *We define the sets*

$$\mathcal{F}_\varphi = \{\, \omega \in A^*_+ \mid \omega \leq \varphi \,\}$$

and

$$\mathcal{G}_\varphi = \{\, \alpha\, \omega \mid \omega \in \mathcal{F}_\varphi \,,\, \alpha \in \,]0,1[\, \} \ .$$

The advantage of $\mathcal{G}_\varphi$ over $\mathcal{F}_\varphi$ is the fact that $\mathcal{G}_\varphi$ is a directed subset of $\mathcal{F}_\varphi$ (under the normal order on A^*_+). A proof of this fact can be found in [18] or [28]. There is also proof for this fact for a slightly more general case in proposition 2.13 of [13].

The most important result concerning lower semi-continuous weights is the following one (proved in [3]).

Theorem 14.2 *We have for every $x \in A^+$ that*

$$\varphi(x) = \sup_{\omega \in \mathcal{F}_\varphi} \omega(x) \ .$$

This can also be stated in terms of the directed set $\mathcal{G}_\varphi$ as follows:

Proposition 14.3 *Consider an element $x \in A^+$. Then*

1. *The element x belongs to $\mathcal{M}_\varphi^+ \Leftrightarrow$ The net $\big(\omega(x)\big)_{\omega \in \mathcal{G}_\varphi}$ is convergent in $\mathbb{R}^+$.*

2. *If x belongs to $\mathcal{M}_\varphi^+$, then the net $\big(\omega(x)\big)_{\omega \in \mathcal{G}_\varphi}$ converges to $\varphi(x)$.*

Then we get immediately that the net $\big(\omega(x)\big)_{\omega \in \mathcal{G}_\varphi}$ converges to $\varphi(x)$ for every $x \in \mathcal{M}_\varphi$.

Any lower semi-continuous weight φ has a natural extension to a weight on $M(A)$. Remember that every $\omega \in A^*$ has a unique extension $\overline{\omega}$ to $M(A)$ which is strictly continuous and we put $\omega(x) = \overline{\omega}(x)$ for every $x \in M(A)$.

Definition 14.4 *We define the weight $\overline{\varphi}$ on $M(A)$ such that*

$$\overline{\varphi}(x) = \sup_{\omega \in \mathcal{F}_\varphi} \omega(x)$$

for every $x \in M(A)^+$.

We define $\overline{\mathcal{M}}_\varphi = \mathcal{M}_{\overline{\varphi}}$ and $\overline{\mathcal{N}}_\varphi = \mathcal{N}_{\overline{\varphi}}$. For any $x \in \overline{\mathcal{M}}_\varphi$, we put $\varphi(x) = \overline{\varphi}(x)$.

Proposition 14.5 *Consider an element $x \in M(A)^+$. Then*

1. *The element x belongs to $\overline{\mathcal{M}}_\varphi^+ \Leftrightarrow$ The net $\big(\omega(x)\big)_{\omega \in \mathcal{G}_\varphi}$ is convergent in $\mathbb{R}^+$.*

2. *If x belongs to $\overline{\mathcal{M}}_\varphi^+$, then the net $\big(\omega(x)\big)_{\omega \in \mathcal{G}_\varphi}$ converges to $\varphi(x)$.*

As a consequence, we have for every $x \in \overline{\mathcal{M}}_\varphi$ that the the net $\big(\omega(x)\big)_{\omega \in \mathcal{G}_\varphi}$ converges to $\varphi(x)$.

Consider a GNS-construction $(H_\varphi, \Lambda_\varphi, \pi_\varphi)$ of a densely defined lower semi-continuous weight φ. Then there is a natural way to get a GNS-construction of $\overline{\varphi}$:

Proposition 14.6 *The mapping $\Lambda_\varphi : A \rightarrow H$ is closable for the strict topology on $M(A)$ and the norm topology on H. We denote this closure by $\overline{\Lambda}_\varphi$. Then $(H_\varphi, \overline{\Lambda}_\varphi, \overline{\pi}_\varphi)$ is a GNS-construction for $\overline{\varphi}$.*

In particular, we have that $D(\overline{\Lambda}_\varphi) = \overline{\mathcal{N}}_\varphi$ and we put $\Lambda_\varphi(x) = \overline{\Lambda}_\varphi(x)$ for every $x \in \overline{\mathcal{N}}_\varphi$.

Now, we will introduce the class of KMS-weights. All weights used in this paper, will belong to this class. For some more details, we refer to [10].

Definition 14.7 *Consider a C^*-algebra A and a weight φ on A, we say that φ is a KMS-weight on A if and only if φ is a densely defined lower semi-continuous weight on A such that there exists a norm-continuous one-parameter group σ on A satisfying the following properties:*

1. *φ is invariant under σ: $\varphi\sigma_t = \varphi$ for every $t \in \mathbb{R}$.*

2. *We have for every $a \in \mathcal{D}(\sigma_{\frac{i}{2}})$ that $\varphi(a^* a) = \varphi(\sigma_{\frac{i}{2}}(a)\sigma_{\frac{i}{2}}(a)^*)$.*

The modular group σ is called a modular group for φ.

If the weight φ is faithful, then the one-parameter group σ is uniquely determined and is called the modular group of σ. This is not the usual definition of a KMS-weight (see [4]), but we prove in [10] that this definition is equivalent with the usual one.

In the next proposition, we formulate some basic properties of KMS-weights. Therefore, let $(H_\varphi, \Lambda_\varphi, \pi_\varphi)$ be a GNS-construction for φ.

Proposition 14.8 *Consider a C*-algebra A, and a KMS-weight φ on A with modular group σ. Then:*

1. *There exists a unique anti-unitary operator J on H_φ such that $J\Lambda_\varphi(x) = \Lambda_\varphi(\sigma_{\frac{i}{2}}(x)^*)$ for every $x \in \mathcal{N}_\varphi \cap \mathcal{D}(\sigma_{\frac{i}{2}})$.*
2. *Let $a \in \mathcal{D}(\sigma_{\frac{i}{2}})$ and $x \in \mathcal{N}_\varphi$. Then xa belongs to $\mathcal{N}_\varphi$ and $\Lambda_\varphi(xa) = J\pi_\varphi(\sigma_{\frac{i}{2}}(a))^* J\,\Lambda_\varphi(x)$.*
3. *Let $a \in \mathcal{D}(\sigma_{-i})$ and $x \in \mathcal{M}_\varphi$. Then ax and $x\sigma_{-i}(a)$ belong to $\mathcal{M}_\varphi$ and $\varphi(ax) = \varphi(x\sigma_{-i}(a))$.*

14.2 The tensor products of weights

It the next part we will quickly say something about the tensor product of two KMS-weights.
Therefore, consider two C*-algebras A and B. Let φ be a KMS-weight on A with modular group σ and ψ a KMS-weight on B with modular group τ.

Definition 14.9 *We define the tensor product weight $\varphi \otimes \psi$ on $A \otimes B$ in such a way that*

$$(\varphi \otimes \psi)(x) = \sup \{ \, (\omega \otimes \theta)(x) \mid \omega \in \mathcal{F}_\varphi, \eta \in \mathcal{F}_\psi \, \}$$

for every $x \in (A \otimes B)^+$. Then $\varphi \otimes \psi$ is a densely defined lower semi-continuous weight on $A \otimes B$ such that $\mathcal{M}_\varphi \odot \mathcal{M}_\psi \subseteq \mathcal{M}_{\varphi\otimes\psi}$ and $(\varphi \odot \psi)(a \otimes b) = \varphi(a)\,\psi(b)$ for $a \in \mathcal{M}_\varphi$ and $b \in \mathcal{M}_\psi$.

Of course, this definition is also possible for lower semi-continuous weights and is done in [18] ans [28].

Consider now a GNS-construction $(H_\varphi, \Lambda_\varphi, \pi_\varphi)$ for φ and a GNS-construction $(H_\psi, \Lambda_\psi, \pi_\psi)$ for ψ.

Proposition 14.10 *The mapping $\Lambda_\varphi \odot \Lambda_\psi : \mathcal{N}_\varphi \odot \mathcal{N}_\psi \to H_\varphi \otimes H_\psi$ is closable and we denote the closure of it by $\Lambda_\varphi \otimes \Lambda_\psi$. Then $(H_\varphi \otimes H_\psi, \Lambda_\varphi \otimes \Lambda_\psi, \pi_\varphi \otimes \pi_\psi)$ is a GNS-construction for $\varphi \otimes \psi$.*

The closability of $\Lambda_\varphi \odot \Lambda_\psi$ is also true for lower semi-continuous weights. However the KMS-condition is used to prove the second part of this proposition (For this, it would also be sufficient that φ and ψ satisfy a weaker condition, the so-called regularity condition, see [18] and [28]).

Using the results of [10], it is not so difficult to prove that $\varphi \otimes \psi$ is a KMS-weight on $A \otimes B$ with a modular group $\sigma \otimes \tau$ defined in such a way that $(\sigma \otimes \tau)_t = \sigma_t \otimes \tau_t$ for every $t \in \mathbb{R}$.

We would also like to mention that all of the results (except the lasr one) are also true in the case of so-called regular C*-valued weights (see section 8 of [13]).

14.3 Slicing with weights

In a last part we say something about slicing with weights. For a more detailed exposition, we refer to [13], [18] and [28]. The main reference is [13]. A lot of the results cannot be found literally in [13], mainly because we are in the very special case where ι is a *-homomorphism.

However, we believe this part of the appendix is readable without knowing the precise results of [13] and that it gives a fairly good idea of the problems in [10].

The reader is encouraged to fill in the details of this appendix, knowing that the most difficult problems are resolved in [10] (mainly in section 7 and section 8).

For the rest of this subsection, we fix two C*-algebras A and B and a KMS-weight φ on B. Let $(H_\varphi, \Lambda_\varphi, \pi_\varphi)$ be a GNS-construction for φ.

Definition 14.11 *We define the map $\iota \otimes \varphi$ from within $(A \otimes B)^+$ into A^+ as follows:*

- *We define the set $\mathcal{M}^+_{\iota\otimes\varphi} = \{\, x \in (A \otimes B)^+ \mid$ the net $\big((\iota \otimes \omega)(x)\big)_{\omega \in \mathcal{G}_\varphi}$ is norm convergent in $A \,\}$*
- *The mapping $\iota \otimes \varphi$ will have as domain the set $\mathcal{M}^+_{\iota\otimes\varphi}$ and for any $x \in \mathcal{M}^+_{\iota\otimes\varphi}$, we have by definition that the net $\big((\iota \otimes \omega)(x)\big)_{\omega \in \mathcal{G}_\varphi}$ converges to $(\iota \otimes \varphi)(x)$.*

It is clear that $\mathcal{M}^+_{\iota\otimes\varphi}$ is a dense hereditary cone in $(A \otimes B)^+$. Furthermore we define the following sets:

1. We define the ideal $\mathcal{N}_{\iota\otimes\varphi} = \{\, x \in A \otimes B \mid x^* x$ belongs to $\mathcal{M}^+_{\iota\otimes\varphi} \,\}$.
2. We define the *-algebra $\mathcal{M}_{\iota\otimes\varphi} = \text{span } \mathcal{M}^+_{\iota\otimes\varphi} = \mathcal{N}^*_{\iota\otimes\varphi} \mathcal{N}_{\iota\otimes\varphi}$.

Of course, there exist a unique linear map ψ from $\mathcal{M}_{\iota\otimes\varphi}$ to A which extend $\iota \otimes \varphi$.
For any $x \in \mathcal{M}_{\iota\otimes\varphi}$, we put $(\iota \otimes \varphi)(x) = \psi(x)$.

It is then clear that $\big((\iota \otimes \omega)(x)\big)_{\omega \in \mathcal{G}_\varphi}$ converges to $(\iota \otimes \varphi)(x)$ for every $x \in \mathcal{M}_{\iota\otimes\varphi}$.

We also have for every $x \in \mathcal{M}_{\iota\otimes\varphi}$ and $\theta \in A^*$ that $(\theta \otimes \iota)(x)$ belongs to $\mathcal{M}_\varphi$ and

$$\varphi\big((\theta \otimes \iota)(x)\big) = \theta\big((\iota \otimes \varphi)(x)\big) \ .$$

We are now going to describe some sort of GNS-construction for $\iota \otimes \varphi$. It is possible to prove that the mapping $\iota \odot \Lambda_\varphi$ from $A \odot \mathcal{N}_\varphi$ into $A \odot H$ is closable (as a mapping from the C*-algebra $A \otimes B$ into the Hilbert-C*-module $A \otimes H$). We define $\iota \otimes \Lambda_\varphi$ to be the closure of this mapping $\iota \odot \Lambda_\varphi$.

It is also possible to prove that $D(\iota \otimes \Lambda_\varphi)$ is a subset of $\mathcal{N}_{\iota\otimes\varphi}$ and that

$$(\iota \otimes \varphi)(y^* x) = \langle (\iota \otimes \Lambda_\varphi)(x), (\iota \otimes \Lambda_\varphi)(y) \rangle$$

for every $x, y \in D(\iota \otimes \Lambda_\varphi)$.

All that is said about $\iota \otimes \varphi$ until now goes through for lower semi-continuous weights (and can be found in [18] and [28]). Because we assumed that φ is also KMS, we have also the rather non-trivial result that $D(\iota \otimes \Lambda_\varphi) = \mathcal{N}_{\iota\otimes\varphi}$. This last result is also true if φ would obey a weaker condition called regularity (see [13]).

We have also that $(\iota \otimes \pi_\varphi)(x)(\iota \otimes \Lambda_\varphi)(a) = (\iota \otimes \Lambda_\varphi)(xa)$ for every $x \in M(A \otimes B)$ and $a \in \mathcal{N}_{\iota\otimes\varphi}$.

It is not difficult to check the following results.

Result 14.12 *Consider an element $a \in M(A)$, then we have the following properties.*

1. *We have for every $x \in \mathcal{N}_{\iota\otimes\varphi}$ that $x(a \otimes 1)$ belongs to $\mathcal{N}_{\iota\otimes\varphi}$ and $(\iota \otimes \Lambda_\varphi)(x)\, a = (\iota \otimes \Lambda_\varphi)(x(a \otimes 1))$.*
2. *We have for every $x \in \mathcal{M}_{io\otimes\varphi}$ that $x(a \otimes 1)$ and $(a \otimes 1)x$ belong to $\mathcal{M}_{\iota\otimes\varphi}$ and that*

$$(\iota \otimes \varphi)(x(a \otimes 1)) = (\iota \otimes \varphi)(x)\, a \qquad \textit{and} \qquad (\iota \otimes \varphi)((a \otimes 1)x) = a\, (\iota \otimes \varphi)(x) \ .$$

For the rest of this section, we have to rely on [13].

We would like to have an extension of $\iota \otimes \varphi$ to $M(A \otimes B)$. This is done in the following way:

Definition 14.13 *We define the map $\overline{\iota \otimes \varphi}$ from within $M(A \otimes B)^+$ into $M(A)^+$ as follows:*

- *We define the set $\overline{\mathcal{M}}^+_{\iota \otimes \varphi} = \{\, x \in M(A \otimes B)^+ \mid$ the net $\big((\iota \otimes \omega)(x) \big)_{\omega \in \mathcal{G}_\varphi}$ is strictly convergent in $M(A) \,\}$.*
- *The mapping $\overline{\iota \otimes \varphi}$ will have as domain the set $\overline{\mathcal{M}}^+_{\iota \otimes \varphi}$ and for any $x \in \overline{\mathcal{M}}^+_{\iota \otimes \varphi}$, we have by definition that the net $\big((\iota \otimes \omega)(x) \big)_{\omega \in \mathcal{G}_\varphi}$ converges strictly to $\big(\overline{\iota \otimes \varphi}\big)(x)$.*

This definition is in fact not entirely correct because it depends on φ and not on $\iota \otimes \varphi$. It is possible to give a definition in terms of the mapping $\iota \otimes \varphi$ and our definition would then be a proposition.

We should also mention that the symbol $\iota \otimes \varphi$ in [13] means something different than is in this paper. In fact, $\iota \otimes \varphi$ in [13] denotes the restriction of our map $\overline{\iota \otimes \varphi}$ to $\overline{\mathcal{M}}^+_{\iota \otimes \varphi} \cap A^+$. This difference is however not fundamental.

The next proposition reveals a nice feature about $\overline{\iota \otimes \varphi}$.

Proposition 14.14 *Consider $x \in M(A \otimes B)^+$. Then x belongs to $\overline{\mathcal{M}}^+_{\iota \otimes \varphi}$ if and only if the net $\big(a^*(\iota \otimes \omega)(x) a \big)_{\omega \in \mathcal{G}_\varphi}$ is norm convergent for every $a \in A$.*

It is clear that $\overline{\mathcal{M}}^+_{\iota \otimes \varphi}$ is a hereditary cone in $M(A \otimes B)^+$. Furthermore we define the following sets:

1. We define the left ideal $\overline{\mathcal{N}}_{\iota \otimes \varphi} = \{\, x \in M(A \otimes B) \mid x^* x$ belongs to $\overline{\mathcal{M}}^+_{\iota \otimes \varphi} \,\}$.
2. We define the *-algebra $\overline{\mathcal{M}}_{\iota \otimes \varphi} = \text{span } \overline{\mathcal{M}}^+_{\iota \otimes \varphi} = \overline{\mathcal{N}}^*_{\iota \otimes \varphi} \overline{\mathcal{N}}_{\iota \otimes \varphi}$.

Of course, there exist a unique linear map $\overline{\psi}$ from $\overline{\mathcal{M}}_{\iota \otimes \varphi}$ to $M(A)$ which extend $\overline{\iota \otimes \varphi}$.
For any $x \in \overline{\mathcal{M}}_{\iota \otimes \varphi}$, we put $(\iota \otimes \varphi)(x) = \overline{\psi}(x)$.

It is then clear that $\big((\iota \otimes \omega)(x) \big)_{\omega \in \mathcal{G}_\varphi}$ converges strictly to $(\iota \otimes \varphi)(x)$ for every $x \in \overline{\mathcal{M}}_{\iota \otimes \varphi}$.

We also have for any $x \in \overline{\mathcal{M}}_{\iota \otimes \varphi}$ and $\theta \in A^*$ that $(\theta \otimes \iota)(x)$ belongs to $\overline{\mathcal{M}}_\varphi$ and

$$\varphi\big((\theta \otimes \iota)(x)\big) = \theta\big((\iota \otimes \varphi)(x)\big) \ .$$

We want also to get something like a GNS-construction for $\overline{\iota \otimes \varphi}$.

Notice that $\mathcal{K}(A, A \otimes H)$ can be turned into a Hilbert-C*-module by defining the scalar product on $\mathcal{K}(A, A \otimes H)$ in such a way that $\langle x, y \rangle = y^* x$ for every $x, y \in \mathcal{K}(A, A \otimes H)$.
We can also define a mapping Υ from $A \otimes H$ into $\mathcal{K}(A, A \otimes H)$ such that $\Upsilon(v)\, a = v\, a$ for every $a \in A$ and $v \in A \otimes H$. Then this mapping is a unitary transformation form $A \otimes H$ to $\mathcal{K}(A, A \otimes H)$.
We will use this mapping Υ to identify $\mathcal{K}(A, A \otimes H)$ with $A \otimes H$.

So we can consider $\iota \otimes \Lambda_\varphi$ as a closed linear mapping from $\mathcal{N}_{\iota \otimes \varphi}$ into $\mathcal{K}(A, A \otimes H)$.

One can prove that $\iota \otimes \Lambda_\varphi$ is closable for the strict topology on $M(A \otimes B)$ and the strong-*-topology on $\mathcal{L}(A, A \otimes H)$ and we denote this closure by $\overline{\iota \otimes \Lambda_\varphi}$.
So $\overline{\iota \otimes \Lambda_\varphi}$ is a strictly-strongly* closed linear map from within $M(A \otimes B)$ into $\mathcal{L}(A, A \otimes H)$.

We can prove (but this is not straightforward) that $\overline{\mathcal{N}}_{\iota \otimes \varphi} = D\big(\overline{\iota \otimes \Lambda_\varphi}\big)$.

For every $x \in \overline{\mathcal{N}}_{\iota\otimes\varphi}$, we define $(\iota \otimes \Lambda_\varphi)(x) = \big(\overline{\iota \otimes \Lambda_\varphi}\big)(x)$, so $(\iota \otimes \Lambda_\varphi)(x) \in \mathcal{L}(A, A \otimes H)$.

We have moreover for every $x, y \in \overline{\mathcal{N}}_{\iota\otimes\varphi}$ that

$$(\iota \otimes \varphi)(y^* x) = (\iota \otimes \Lambda_\varphi)(y)^* (\iota \otimes \Lambda_\varphi)(x)$$

We have also that $(\iota \otimes \pi_\varphi)(x)(\iota \otimes \Lambda_\varphi)(a) = (\iota \otimes \Lambda_\varphi)(xa)$ for every $x \in M(A \otimes B)$ and $a \in \overline{\mathcal{N}}_{\iota\otimes\varphi}$.

It is not difficult to check the following results.

Result 14.15 *Consider an element $a \in M(A)$, then we have the following properties.*

1. *We have for every $x \in \overline{\mathcal{N}}_{\iota\otimes\varphi}$ that $x(a \otimes 1)$ belongs to $\overline{\mathcal{N}}_{\iota\otimes\varphi}$ and $(\iota \otimes \Lambda_\varphi)(x)\, a = (\iota \otimes \Lambda_\varphi)(x(a \otimes 1))$.*
2. *We have for every $x \in \overline{\mathcal{M}}_{io\otimes\varphi}$ that $x(a \otimes 1)$ and $(a \otimes 1)x$ belong to $\overline{\mathcal{M}}_{\iota\otimes\varphi}$ and that*

$$(\iota \otimes \varphi)(x(a \otimes 1)) = (\iota \otimes \varphi)(x)\, a \qquad \text{and} \qquad (\iota \otimes \varphi)((a \otimes 1)x) = a\, (\iota \otimes \varphi)(x)\ .$$

We would like to end this part with some remarks

- Consider $x \in \overline{\mathcal{N}}_{\iota\otimes\varphi}$, then $(\iota \otimes \varphi)(x^* x)$ belongs to A if and only $(\iota \otimes \Lambda_\varphi)(x)$ belongs to $\mathcal{K}(A, A \otimes H)$.
- Consider $x \in (A \otimes B)^+$, then x belongs to $\mathcal{M}^+_{\iota\otimes\varphi}$ if and only if x belongs to $\overline{\mathcal{M}}^+_{\iota\otimes\varphi}$ and $(\iota \otimes \varphi)(x)$ belongs to A^+.

We are also interested in extending $\iota \otimes \varphi$ even further and let it take values in the set of elements affiliated with A.

Let $x \in M(A \otimes B)$ and define the set

$$D_x = \{\, a \in A \mid x(a \otimes 1) \in \overline{\mathcal{N}}_{\iota\otimes\varphi} \text{ and } (\iota \otimes \varphi)((a^* \otimes 1)x^* x(a \otimes 1)) \in A \,\}$$

Then we have for every $a \in A$ that a belongs to D_x if and only if the net $\big(a^*\, (\iota \otimes \omega)(x^* x)\, a\big)_{\omega \in \mathcal{G}_\varphi}$ is convergent in A.

Definition 14.16 *We define the set*

$$\tilde{\mathcal{N}}_{\iota\otimes\varphi} = \{\, x \in M(A \otimes B) \mid D_x \textit{ is dense in } A \,\}$$

Definition 14.17 *Consider $x \in \tilde{\mathcal{N}}_{\iota\otimes\varphi}$. Then we define the mapping $(\iota \otimes \Lambda_\varphi)(x)$ from D_x into $A \otimes H$ as follows : Let $a \in D_x$. Then we know that $x(a \otimes 1)$ belongs to $\overline{\mathcal{N}}_{\iota\otimes\varphi}$ and $(\iota \otimes \Lambda_\varphi)(x(a \otimes 1))$ belongs to $\mathcal{K}(A, A \otimes H)$. So there exists a unique element $v \in A \otimes H$ such that $(\iota \otimes \Lambda_\varphi)(x(a \otimes 1))\, b = v\, b$ for every $b \in A$ and we define $(\iota \otimes \Lambda_\varphi)(x)(a) = v$.*

It is then possible to prove that $(\iota \otimes \Lambda_\varphi)(x)$ is a closed densely defined linear operator from within A into $A \otimes H$ such that $(\iota \otimes \Lambda_\varphi)(x)^*$ is densely defined.

Definition 14.18 *We define the set*

$$\hat{\mathcal{N}}_{\iota\otimes\varphi} = \{\, x \in \tilde{\mathcal{N}}_{\iota\otimes\varphi} \mid (\iota \otimes \Lambda_\varphi)(x) \textit{ is regular } \}\ .$$

Definition 14.19 *We define the set*

$$\hat{\mathcal{M}}^+_{\iota\otimes\varphi} = \{\, x \in M(A \otimes B)^+ \mid x^{\frac{1}{2}} \in \hat{\mathcal{N}}_{\iota\otimes\varphi} \ .$$

For every $x \in \hat{\mathcal{M}}^+_{\iota\otimes\varphi}$, we put $(\iota\otimes\varphi)(x) = (\iota\otimes\Lambda_\varphi)(x^{\frac{1}{2}})^*(\iota\otimes\Lambda_\varphi)(x^{\frac{1}{2}})$, so $(\iota\otimes\varphi)(x)$ is a positive element affiliated with A.

Consider $y \in M(A\otimes B)$. Then one can prove that $y \in \hat{\mathcal{N}}_{\iota\otimes\varphi}$ if and only if $y^*y \in \hat{\mathcal{M}}^+_{\iota\otimes\varphi}$ and we have in this case that $(\iota\otimes\varphi)(y^*y) = (\iota\otimes\Lambda_\varphi)(y)^*(\iota\otimes\Lambda_\varphi)(y)$

It goes without saying that everything goes also through for the slicing $\varphi\otimes\iota$.

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